Exercise Sheet 6

This sheet is due in the lecture on Tuesday 11th November, and will be discussed in the exercise class on Friday 14th November.

Exercise 6.1. *Examples of algebraic integers.*

(1) Show that $\frac{1}{2}(1+\sqrt{5})$ is an algebraic integer by definition; i.e. by writing down a monic polynomial in $\mathbb{Z}[x]$ for which it is a root. Do the same for $3+i$ and $\sqrt{2}+\sqrt[3]{3}$.

(2) Show that $\frac{1}{2}$ is an algebraic number but not an algebraic integer by definition.

(3) Suppose that α is an algebraic integer. Show that $-\alpha$ is also an algebraic integer.

Exercise 6.2. *Examples of traces and norms.*

(1) Let K be the cubic field $\mathbb{Q}(\sqrt[3]{2})$. For any $\alpha = a + b\sqrt[3]{2} + c\sqrt[3]{4} \in K$ with $a, b, c \in \mathbb{Q}$, write down the matrix for the linear transformation L_α under the basis $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$. Compute the trace and norm of α in K.

(2) Let K be the cyclotomic field $\mathbb{Q}(\zeta)$ where $\zeta = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Write down the matrix for the linear transformation L_ζ under the basis $\{1, \zeta, \zeta^2, \zeta^3\}$. Compute the trace and norm of ζ in K.

Exercise 6.3. *Elementary properties of the trace and norm.*

Let K be a number field of degree n over $\mathbb{Q}$, $\alpha, \beta \in K$ and $a \in \mathbb{Q}$. Prove the following

(1) $T(\alpha+\beta) = T(\alpha) + T(\beta)$, $N(\alpha\beta) = N(\alpha)N(\beta)$;

(2) $T(a\alpha) = aT(\alpha)$, $N(a\beta) = a^n N(\beta)$;

(3) $T(1) = n$, $N(1) = 1$;

(4) $N(\alpha) = 0$ iff $\alpha = 0$.

Exercise 6.4. *Traces and norms of algebraic integers.*

Supply the details in the proof of Proposition 6.19 in the following steps. The set S is defined in the sketch of proof in the lecture notes.

(1) Show that S spans K over $\mathbb{Q}$, i.e. every element in K is a $\mathbb{Q}$-linear combination of elements in S with rational coefficients;

(2) Show that elements in S are linearly independent over $\mathbb{Q}$;

(3) Write down the matrix for L_α under the basis S. Conclude that all entries are in $\mathbb{Z}$, and $T(\alpha), N(\alpha) \in \mathbb{Z}$.