Exercise Sheet 7

This sheet is due in the lecture on Tuesday 18th November, and will be discussed in the exercise class on Friday 21st November.

Exercise 7.1. *Examples of discriminants.*

(1) Let K be the cubic field $\mathbb{Q}(\sqrt[3]{2})$. Compute the discriminant $\Delta(1, \sqrt[3]{2}, \sqrt[3]{4})$.

(2) Let K be the cyclotomic field $\mathbb{Q}(\zeta)$ where $\zeta = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Compute the discriminant $\Delta(1, \zeta, \zeta^2, \zeta^3)$.

Exercise 7.2. *The discriminant of an ideal is independent of the choice of integral basis.*

Supply the proof of Lemma 7.11 in the following steps.

(1) Show that there exist $n \times n$ matrices M and N with integer entries, such that

$$\Delta(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\det M)^2 \Delta(\beta_1, \beta_2, \cdots, \beta_n),$$
$$\Delta(\beta_1, \beta_2, \cdots, \beta_n) = (\det N)^2 \Delta(\alpha_1, \alpha_2, \cdots, \alpha_n).$$

(2) Show that $(\det M)^2(\det N)^2 = 1$. Conclude that $(\det M)^2 = (\det N)^2 = 1$ and $\Delta(\alpha_1, \alpha_2, \cdots, \alpha_n) = \Delta(\beta_1, \beta_2, \cdots, \beta_n)$.

Exercise 7.3. *The discriminant of a quadratic field.*

Supply the proof of Proposition 7.14 in the following two cases.

(1) Suppose $d \neq 1$ is a square-free integer, $d \equiv 2$ or $3 \pmod 4$ and $K = \mathbb{Q}(\sqrt{d})$. Compute $\Delta(1, \sqrt{d})$. What is the value for Δ_K in this case?

(2) Suppose $d \neq 1$ is a square-free integer, $d \equiv 1 \pmod 4$ and $K = \mathbb{Q}(\sqrt{d})$. Compute $\Delta(1, \frac{1+\sqrt{d}}{2})$. What is the value for Δ_K in this case?

Exercise 7.4. *Integral basis for a principal ideal.*

Let K be a number field of degree n over $\mathbb{Q}$. Assume $\{\omega_1, \omega_2, \cdots, \omega_n\}$ is an integral basis for $\mathcal{O}_K$. Let $\alpha \in \mathcal{O}_K$, $\alpha \neq 0$ and $I = (\alpha)$. Show that $\{\alpha\omega_1, \alpha\omega_2, \cdots, \alpha\omega_n\}$ is an integral basis for I in the following steps.

(1) Show that $\alpha\omega_i \in I$ for each i, $1 \leqslant i \leqslant n$.

(2) Show that $\{\alpha\omega_1, \alpha\omega_2, \cdots, \alpha\omega_n\}$ are $\mathbb{Q}$-linearly independent. Conclude that it is a $\mathbb{Q}$-basis for K.

(3) Show that every $\gamma \in I$ is a linear combination of elements in $\{\alpha\omega_1, \alpha\omega_2, \cdots, \alpha\omega_n\}$ with integer coefficients. Conclude that it is an integral basis for I.

(4) As an example, suppose $K = \mathbb{Q}(\sqrt{3})$. Let $\alpha = \sqrt{3}$ and $I = (\alpha)$ an ideal in $\mathcal{O}_K$. Write down an integral basis for I, and use it to compute the discriminant $\Delta(I)$.