Exercise Sheet 8

This sheet is due in the lecture on Tuesday 25th November, and will be discussed in the exercise class on Friday 28th November.

Exercise 8.1. *Examples of norms of ideals.*

(1) Let $d \neq 1$ be a square-free integer and $K = \mathbb{Q}(\sqrt{d})$. For any algebraic integer $\alpha = a + b\sqrt{d} \in \mathcal{O}_K$, let $I = (\alpha)$. Find the norm $N(I)$. (Hint: Proposition 8.9.)

(2) Let K be a number field of degree n over $\mathbb{Q}$, $a \in \mathbb{Z}$. Let $I = (a)$ be the principal ideal in $\mathcal{O}_K$ generated by a. Find the norm $N(I)$. (Hint: Proposition 8.9.)

Exercise 8.2. *Examples of sums and products of ideals.*

Let R be a commutative ring with 1.

(1) Let I and J be ideals in R. Show that $IJ \subseteq I$ and $I \subseteq I + J$.

(2) Let I be an ideal in R, $\alpha \in R$. Show that $(\alpha)I = \{\alpha\gamma \mid \gamma \in I\}$.

(3) Let $\alpha, \beta \in R$. Show that $(\alpha)(\beta) = (\alpha\beta)$.

(4) Let $\Bbbk$ be a field. The ideal (x, y) in $\Bbbk[x, y]$ is defined to be the sum of the two principal ideals $(x) + (y)$. Show that (x, y) consists of all polynomials in $\Bbbk[x, y]$ whose constant terms are 0.

Exercise 8.3. *Examples of prime and maximal ideals.*

(1) Let $p \in \mathbb{Z}$ be prime. Show that the principal ideal (p) in $\mathbb{Z}$ is prime and maximal.

(2) Let $\Bbbk$ be a field. Show that the principal ideal (x) in $\Bbbk[x]$ is prime and maximal.

(3) Let $\Bbbk$ be a field. Show that the ideal (x, y) in $\Bbbk[x, y]$ is prime and maximal. Show that the principal ideal (x) in $\Bbbk[x, y]$ is prime but not maximal.

Exercise 8.4. *Cancellation law and "to contain is to divide".*

(1) Prove Corollary 8.14. (Hint: by Proposition 8.13, there is an ideal J such that $IJ = (\gamma)$ is a non-zero principal ideal. Multiply $IJ_1 = IJ_2$ on both sides by J to get $(\gamma)J_1 = (\gamma)J_2$. Then show that $J_1 \subseteq J_2$ and similarly $J_2 \subseteq J_1$ to conclude.)

(2) Prove Corollary 8.15. (Hint: first explain why the statement is clear if $I_2 = 0$. If $I_2 \neq 0$, then by Proposition 8.13, there is an ideal I_3 and $\gamma \neq 0$ such that $I_2I_3 = (\gamma)$. Hence we have $I_1I_3 \subseteq I_2I_3 = (\gamma)$. Define the set $J = \{\alpha \in \mathcal{O}_K \mid \gamma\alpha \in I_1I_3\}$. Show that J is an ideal, and that $I_1I_3 = (\gamma)J = I_2I_3J$. Then apply the cancellation law proved in part (1) to conclude $I_1 = I_2J$.)