Exercise Sheet 9

This sheet is due in the lecture on Tuesday 2nd December, and will be discussed in the exercise class on Friday 5th December.

Exercise 9.1. *Card games and non-card games.*

Answer the following questions. You do not need to justify your answers.

(1) Which of the following shape(s) is/are convex? (i) a spade; (ii) a heart; (iii) a club; (iv) a diamond; (v) a joker. (Hint: think of these shapes as in a standard 52-card deck, but pretend that the four sides of the diamond are straight line segments.)

(2) Which of the following shape(s) is/are centrally symmetric? (i) a square with vertices $(0,0),(1,0),(1,1),(0,1)$; (ii) a rhombus with vertices $(1,0)$, $(0,2)$, $(-1,0)$, $(0,-2)$; (iii) a triangle with vertices $(1,-1),(0,1),(-1,-1)$; (iv) a parallelogram with vertices $(2,3)$, $(3,4)$, $(-2,-3)$, $(-3,-4)$; (v) a disk $\{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + (y-1)^2 \leqslant 1\}$; (vi) an annulus $\{(x,y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 2\}$.

Exercise 9.2. *Applications of Minkowski's Theorem.*

(1) Assume we have a lattice L of rank 2 in $\mathbb{R}^2$ whose fundamental domain has volume A. For which positive values of r is the disk $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant r^2\}$ guaranteed to contain at least one non-zero point of L?

(2) Assume we have a lattice L of rank 2 in $\mathbb{R}^2$ whose fundamental domain has volume A. For which positive values of r is the square $S = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leqslant r\}$ guaranteed to contain at least one non-zero point of L?

Exercise 9.3. *Basic properties of ideal classes.*

(1) Show that the relation $\sim$ in Definition 9.1 is an equivalence relation. (Hint: an equivalence relation requires (i) reflexivity: $I \sim I$; (ii) symmetry: if $I \sim J$ then $J \sim I$; (iii) transitivity: if $I_1 \sim I_2$ and $I_2 \sim I_3$ then $I_1 \sim I_3$.)

(2) Show that the product of ideal classes is well-defined; i.e. if $I_1 \sim I_2$ and $J_1 \sim J_2$, then $I_1 J_1 \sim I_2 J_2$.

Exercise 9.4. *Volume of the fundamental domain for real quadratic fields.*

Supply the proof of Proposition 9.14 in the following steps.

(1) Prove L_I is a lattice of rank 2 in $\mathbb{R}^2$ by writing down a pair of generators e_1, e_2.

(2) Use the integral basis of $\mathcal{O}_K$ given in Proposition 7.2 to compute $\mathrm{vol}(T_{\mathcal{O}_K})$.

(3) Use a matrix M to relate $\mathrm{vol}(T_I)$ and $\mathrm{vol}(T_{\mathcal{O}_K})$, and prove the formula for $\mathrm{vol}(T_I)$.