EXERCISE SHEET 10

This sheet is NOT due in the lecture on Tuesday 9th December, and will be discussed in an exercise class in the same week.

Exercise 10.1. Some computation of class numbers.

- (1) Compute the class number of $K = \mathbb{Q}(\sqrt{2})$.
- (2) Compute the class number of $K = \mathbb{Q}(\sqrt{6})$. (Hint: what is norm of $I = (2 + \sqrt{6})$?)
- (3) Compute the class number of $K = \mathbb{Q}(\sqrt{-13})$. (Hint: at some point you need to explain why the only ideal of norm 4 is the principal ideal (2).)

Exercise 10.2. Fermat's two square problem (revisited).

Let p be a positive prime such that $p \equiv 1 \pmod{4}$.

- (1) Show that there exists some $u \in \mathbb{Z}$, such that $u^2 + 1 \equiv 0 \pmod{p}$.
- (2) Consider the lattice $L = \{m_1e_1 + m_2e_2 \mid m_1, m_2 \in \mathbb{Z}\}$ where $e_1 = (1, u)$ and $e_2 = (0, p)$. Compute the volume of its fundamental domain.
- (3) Show that the disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \frac{3}{2}p\}$ contains at least one non-zero point $(a, b) \in L$. Show that $0 < a^2 + b^2 < 2p$.
- (4) Use the generators of L to show $b \equiv ua \pmod{p}$, then show $a^2 + b^2 \equiv 0 \pmod{p}$.
- (5) Conclude from parts (3) and (4) that $p = a^2 + b^2$.

Exercise 10.3. Minkowski bound for real quadratic fields.

Supply the proofs of Propositions 10.4 and 10.5 in the following steps.

- (1) For any real numbers $x, y \in \mathbb{R}$, show that $|xy| \leq \frac{1}{4}(|x| + |y|)^2$.
- (2) Let $r = \left(2N(I) |\Delta_K|^{\frac{1}{2}}\right)^{\frac{1}{2}}$. Show that the square $S = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq r\}$ contains at least one non-zero point in L_I . (Hint: Proposition 9.14.)
- (3) Reinterpret the result in part (2) as follows: there exists some non-zero $\alpha = a + b\sqrt{d} \in I$, such that for $x = a + b\sqrt{d}$ and $y = a b\sqrt{d}$, we have $|x| + |y| \leq r$.
- (4) Use parts (1) and (3) to show that $|N(\alpha)| = |xy| \leq \frac{1}{4}r^2 = \frac{1}{2}N(I) |\Delta_K|^{\frac{1}{2}}$. This proves Proposition 10.4.
- (5) Prove Proposition 10.5. (Hint: almost identical to the proof of Proposition 10.3.)

Exercise 10.4. Review and reinforce knowledge.

If you have any questions, or you want to see more examples of anything in this course, please write them down.