1.2. Arithmetic functions. An arithmetic function is a complex valued function defined on the set of positive integers, or in other words, simply a sequence of complex numbers.

Definition 1.16. An arithmetic function is a function

$$
f: \mathbb{Z}^{+} \rightarrow \mathbb{C}
$$

where $\mathbb{Z}^{+}$is the set of all positive integers.

In principle one could assign any complex number as the value of the function at an positive integer. We look at some examples.

Example 1.17. Here are some very simple examples.
(1) For every complex number $c \in \mathbb{C}$, we can define the constant function

$$
f_{c}: \mathbb{Z}^{+} \rightarrow \mathbb{C} \quad \text { given by } \quad f_{c}(n)=c \quad \text { for every } n \in \mathbb{Z}^{+} .
$$

In particular, we denote the function which takes constant values 1 by $I$.
(2) Another function which will show up later will be the function $\mathbb{I}$ defined by

$$
\mathbb{I}(n)= \begin{cases}1 & \text { if } n=1 \\ 0 & \text { if } n>1\end{cases}
$$

However we are mainly interested in arithmetic functions with a meaningful assignment of values, most of which take values in integers.

Example 1.18. Here are some naturally defined arithmetic functions.

- For any $n \in \mathbb{Z}^{+}$, define $\nu(n)$ to be the number of positive divisors of $n$;
- For any $n \in \mathbb{Z}^{+}$, define $\sigma(n)$ to be the sum of the positive divisors of $n$.

By virtue of the unique factorisation, we can obtain the following formulas for the two functions:

Proposition 1.19. Assume the integer $n>1$ has the prime decomposition

$$
n=p_{1}^{a_{1}} p_{2}^{a_{2}} \cdots p_{l}^{a_{l}},
$$

where $p_{1}, p_{2}, \cdots, p_{l}$ are distinct positive primes. Then we have

$$
\begin{aligned}
\nu(n) & =\left(a_{1}+1\right)\left(a_{2}+1\right) \cdots\left(a_{l}+1\right) ; \\
\sigma(n) & =\frac{p_{1}^{a_{1}+1}-1}{p_{1}-1} \cdot \frac{p_{2}^{a_{2}+1}-1}{p_{2}-1} \cdots \frac{p_{l}^{a_{l}+1}-1}{p_{l}-1} .
\end{aligned}
$$

Proof. To prove the first formula, we notice that $m \mid n$ iff

$$
m=p_{1}^{b_{1}} p_{2}^{b_{2}} \cdots p_{l}^{b_{l}}
$$

with $0 \leqslant b_{i} \leqslant a_{i}$ for every $i$. Thus the positive divisors of $n$ are one-to-one correspondent to the $n$-tuples ( $b_{1}, b_{2}, \cdots, b_{l}$ ) with $0 \leqslant b_{i} \leqslant a_{i}$ for every $i$, and there are exactly

$$
\left(a_{1}+1\right)\left(a_{2}+1\right) \cdots\left(a_{l}+1\right)
$$

such $n$-tuples.
To prove the second formula, we notice that

$$
\sigma(n)=\sum_{1 \leqslant b_{1} \leqslant a_{1}, 1 \leqslant b_{2} \leqslant a_{2}, \cdots, 1 \leqslant b_{l} \leqslant a_{l}} p_{1}^{b_{1}} p_{2}^{b_{2}} \cdots p_{l}^{b_{l}}
$$

where the sum is over the above set of $n$-tuples. Thus we can see that

$$
\sigma(n)=\left(1+p_{1}+p_{1}^{2}+\cdots+p_{1}^{a_{1}}\right)\left(1+p_{2}+p_{2}^{2}+\cdots+p_{2}^{a_{2}}\right) \cdots\left(1+p_{l}+p_{l}^{2}+\cdots+p_{l}^{a_{l}}\right)
$$

from which the result follows by applying the summation formula for geometric series.

Next example is another arithmetic function which will play an important role in Möbius inversion theorem. For convenience, we say an integer $n$ square-free if it is not divisible by the square of any integer greater than 1 . An equivalent characterisation: $n$ is square-free iff $n$ does not have repeated prime factors in its prime decomposition. In other words, $n$ is square-free iff $n$ is the product of finitely many distinct primes.

Definition 1.20. For any positive integer $n$, we define the Möbius $\mu$-function by

$$
\mu(n)= \begin{cases}1 & \text { if } n=1 \\ 0 & \text { if } n \text { is not square-free; } \\ (-1)^{l} & \text { if } n=p_{1} p_{2} \cdots p_{l} \text { is the product of } l \text { distinct primes }\end{cases}
$$

We prove the following property of Möbius $\mu$-function. Again, the unique factorisation is the key to the proof.

Proposition 1.21. For any $n \in \mathbb{Z}^{+}$, we have

$$
\sum_{d \mid n} \mu(d)= \begin{cases}1 & \text { if } n=1 \\ 0 & \text { if } n>1,\end{cases}
$$

where the summation runs over all positive divisors of $n$.
Proof. The case of $n=1$ is clear. Now we assume $n \geqslant 2$. Let $n=p_{1}^{a_{1}} p_{2}^{a_{2}} \cdots p_{l}^{a_{l}}$ be the prime decomposition of $n$ for some $l \in \mathbb{Z}^{+}$. The definition of $\mu$-function shows that only those divisors $d$ of $n$ which do not have repeated prime factors contribute to the summation. For any $i$ with $0 \leqslant i \leqslant l$, we consider the number of divisors $d$ of $n$ which
are products of $i$ distinct primes. Since the prime factors of $d$ form a subset of those of $n$, there are exactly $\binom{l}{i}$ choices for such $d$, each of which contributes $(-1)^{i}$ to $\mu(n)$. Therefore we have

$$
\begin{aligned}
\sum_{d \mid n} \mu(d) & =\binom{l}{0}-\binom{l}{1}+\binom{l}{2}-\binom{l}{3}+\cdots+(-1)^{l}\binom{l}{l} \\
& =(1-1)^{l}=0 .
\end{aligned}
$$

The definition of the $\mu$-function seems somewhat artificial at the first glance. However its significance will not be revealed until we introduce Dirichlet products of arithmetic functions.

