9.2. **Minkowski's theorem.** From now on we will focus on quadratic fields $\mathbb{Q}(\sqrt{d})$ for any square-free integer d and prove their class numbers are finite; see Example 6.12 (1). Now we state the famous Minkowski's Theorem in dimension 2, which is the main tool in studying this problem.

Theorem 9.11 (Minkowski's Theorem)**.** *Let L be a lattice of rank 2 in $\mathbb{R}^2$ with fundamental domain T. Let X be a centrally symmetric convex subset of $\mathbb{R}^2$. If $\mathrm{vol}(X) > 4\,\mathrm{vol}(T)$, then X contains a non-zero point of L.*

Proof. We first shrink X to half of its size in length; precisely speaking, we consider $Y = \{p \in \mathbb{R}^2 \mid 2p \in X\}$. Then $\mathrm{vol}(Y) = \frac{1}{4}\mathrm{vol}(X) > \mathrm{vol}(T)$.

For every $h \in L$, we define $h + T = \{h + p \mid p \in T\}$ which is the transport of the fundamental domain along the vector h. It is clear that $\mathbb{R}^2$ becomes the disjoint union of these parallelograms. Let $Y_h = Y \cap (h+T)$ is the part of Y which lies in the parallelogram $h+T$ for each $h \in L$, then Y becomes the disjoint union of all Y_h's, hence $\sum_{h\in L}\mathrm{vol}(Y_h) = \mathrm{vol}(Y) > \mathrm{vol}(T)$. We transport each Y_h back to the fundamental domain, say $Y'_h = \{q \in T \mid h + q \in Y_h\}$. Then $\sum_{h\in L}\mathrm{vol}(Y'_h) = \sum_{h\in L}\mathrm{vol}(Y_h) > \mathrm{vol}(T)$. Since each $Y'_h \subseteq T$, this inequality implies they are not disjoint. Therefore there exist $h_1, h_2 \in L$, $h_1 \neq h_2$, such that we can find some $q \in Y'_{h_1} \cap Y'_{h_2}$. That implies $p_1 = h_1 + q \in Y_{h_1} \subseteq Y$ and $p_2 = h_2 + q \in Y_{h_2} \subseteq Y$, hence we found $p_1, p_2 \in Y$, such that $p_1 - p_2 = h_1 - h_2 \in L$.

Since $p_1, p_2 \in Y$, we have $2p_1, 2p_2 \in X$. Since X is centrally symmetric, $-2p_2 \in X$. Since X is convex, $\frac{1}{2}(2p_1) + \frac{1}{2}(-2p_2) \in X$. And $\frac{1}{2}(2p_1) + \frac{1}{2}(-2p_2) = h_1 - h_2$ is a non-zero point in L. ☐

Corollary 9.12. *Let L be a lattice of rank 2 in $\mathbb{R}^2$ with fundamental domain T. Let X be a centrally symmetric convex subset of $\mathbb{R}^2$. If X is compact (i.e. closed and bounded), and $\mathrm{vol}(X) \geqslant 4\,\mathrm{vol}(T)$, then X contains a non-zero point of L.*

Proof. We do not prove this corollary rigorously because it requires some understanding of topology. Intuitively, we can enlarge X a little bit so that we can apply Theorem 9.11 and obtain lattice points in the enlarged X. Since this enlargement can be arbitrarily tiny, there must be lattice points within the boundary of X. ☐

As an indication on how geometry can be used to study number fields, we construct lattices from some familiar objects. Here we consider a quadratic number field $K = \mathbb{Q}(\sqrt{d})$ for any square-free integer d. As usual, its ring of integers is denoted by $\mathcal{O}_K$ and let I be any non-zero ideal of $\mathcal{O}_K$.

Proposition 9.13. *Let $d < 0$ be a square-free integer and $K = \mathbb{Q}(\sqrt{d})$ a quadratic field. For any non-zero ideal I in $\mathcal{O}_K$, the set $L_I = \{(\mathrm{Re}\,\alpha, \mathrm{Im}\,\alpha) \in \mathbb{R}^2 \mid \alpha \in I\}$ is a lattice of rank 2 in $\mathbb{R}^2$. Let T_I be the fundamental domain of L_I, then $\mathrm{vol}(T_I) = \frac{1}{2}N(I)\,|\Delta_K|^{\frac{1}{2}}$.*

Proof. This proposition can be proved in three steps.

Step 1. We prove that L_I is a lattice of rank 2 in $\mathbb{R}^2$. By Proposition 7.9, assume α_1, α_2 is an integral basis for I, then we can write $I = \{m_1\alpha_1 + m_2\alpha_2 \mid m_1, m_2 \in \mathbb{Z}\}$. Let $e_1 = (\operatorname{Re}\alpha_1, \operatorname{Im}\alpha_1)$ and $e_2 = (\operatorname{Re}\alpha_2, \operatorname{Im}\alpha_2)$, then for every $\alpha = m_1\alpha_1 + m_2\alpha_2 \in I$, $(\operatorname{Re}\alpha, \operatorname{Im}\alpha) = m_1(\operatorname{Re}\alpha_1, \operatorname{Im}\alpha_1) + m_2(\operatorname{Re}\alpha_2, \operatorname{Im}\alpha_2) = m_1e_1 + m_2e_2$. Hence $L_I = \{m_1e_1 + m_2e_2 \mid m_1, m_2 \in \mathbb{Z}\}$ is a rank 2 lattice in $\mathbb{R}^2$.

Step 2. We calculate the volume of the fundamental domain in a special case, i.e. $T_{\mathcal{O}_K}$. By Proposition 7.2, we can write $\mathcal{O}_K = \{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}\}$, where $\omega_1 = 1$, and $\omega_2 = \sqrt{d}$ if $d \equiv 2$ or $3 \pmod 4$ and $\frac{1}{2}(1+\sqrt{d})$ if $d \equiv 1 \pmod 4$.

When $d \equiv 2$ or $3 \pmod 4$, we have $e_1 = (\operatorname{Re}\omega_1, \operatorname{Im}\omega_1) = (1, 0)$ and $e_2 = (\operatorname{Re}\omega_2, \operatorname{Im}\omega_2) = (0, \sqrt{-d})$. Hence the volume of the fundamental domain is

$$\operatorname{vol}(T_{\mathcal{O}_K}) = \left|\det\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-d}\end{pmatrix}\right| = \sqrt{-d} = \frac{1}{2}\left|\Delta_K\right|^{\frac{1}{2}},$$

where the last equality follows from Proposition 7.14.

When $d \equiv 1 \pmod 4$, we have $e_1 = (\operatorname{Re}\omega_1, \operatorname{Im}\omega_1) = (1, 0)$ and $e_2 = (\operatorname{Re}\omega_2, \operatorname{Im}\omega_2) = (\frac{1}{2}, \frac{1}{2}\sqrt{-d})$. Hence the volume of the fundamental domain is

$$\operatorname{vol}(T_{\mathcal{O}_K}) = \left|\det\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2}\sqrt{-d}\end{pmatrix}\right| = \frac{1}{2}\sqrt{-d} = \frac{1}{2}\left|\Delta_K\right|^{\frac{1}{2}},$$

where the last equality still follows from Proposition 7.14.

Step 3. We calculate the volume of the fundamental domain T_I in general. For an arbitrary ideal I with an integral basis α_1, α_2, we can write $\alpha_1 = a_{11}\omega_1 + a_{21}\omega_2$ and $\alpha_2 = a_{12}\omega_1 + a_{22}\omega_2$, as well as the transition matrix $M = (a_{ij})$, where $a_{ij} \in \mathbb{Z}$. By taking real parts and imaginary parts of α_1 and α_2, we realise that they can be organised into the following matrix

$$\begin{pmatrix} \operatorname{Re}\alpha_1 & \operatorname{Re}\alpha_2 \\ \operatorname{Im}\alpha_1 & \operatorname{Im}\alpha_2\end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} \operatorname{Re}\omega_1 & \operatorname{Re}\omega_2 \\ \operatorname{Im}\omega_1 & \operatorname{Im}\omega_2\end{pmatrix}.$$

Taking determinants and absolute values on both sides, we get

$$\operatorname{vol}(T_I) = |\det M| \operatorname{vol}(T_{\mathcal{O}_K}).$$

By Proposition 8.3 and step 2, we conclude that

$$\operatorname{vol}(T_I) = \frac{1}{2}N(I)\left|\Delta_K\right|^{\frac{1}{2}}$$

as required. □

A parallel statement can be established as follows

Proposition 9.14. *Let $d > 1$ be square-free and $K = \mathbb{Q}(\sqrt{d})$ a quadratic field. For any non-zero ideal I of $\mathcal{O}_K$, the set $L_I = \left\{(a + b\sqrt{d}, a - b\sqrt{d}) \in \mathbb{R}^2 \mid a + b\sqrt{d} \in I, a, b \in \mathbb{Q}\right\}$ is a lattice of rank 2 in $\mathbb{R}^2$. Let T_I be the fundamental domain of L_I, then* $\mathrm{vol}(T_I) = N(I)\,|\Delta_K|^{\frac{1}{2}}$.

Proof. We leave it as an exercise. See Exercise 9.4. $\square$