EXTRA HINTS FOR EXERCISE SHEET 6

Exercise 6.1. For part (1), follow the examples from lecture. For part (2), use contradiction. Assume $\frac{1}{2}$ is the root of a monic polynomial with integer coefficients. Then $(\frac{1}{2})^n + a_1(\frac{1}{2})^{n-1} + \cdots + a_n = 0$ for some integers $a_1, \cdots, a_n$. Clear the denominators and find the contradiction! For part (3), if α is the root of a monic polynomial $f(x)$ with integer coefficients, how can you write down a polynomial for which $-\alpha$ is a root?

Exercise 6.2. You need to remember how to write down the matrix for a linear transformation under a basis. The matrix in part (2) is 4×4 but quite sparse, so the determinant is not difficult to compute. Be careful of the sign though.

Exercise 6.3. As long as you interpret every statement in terms of linear transformations or matrices, they should be mostly straightforward. One direction in part (4) is clear. For the other direction, notice that $N(\alpha) = 0$ means the determinant of the matrix for L_α is 0, hence it has a non-trivial null space. Pick any non-zero vector in the null space, which corresponds to a non-zero element in K, say, γ. What does it mean by $L_\alpha(\gamma) = 0$?

Exercise 6.4. Part (1): for any $\gamma \in K$, since K is a vector space over $\mathbb{Q}(\alpha)$, we can write γ as a linear combination of β_j's, with coefficients in $\mathbb{Q}(\alpha)$. Each of such coefficients is a linear combination of α^i's.

Part (2): to show independence, assume a certain linear combination of $\alpha^i\beta_j$'s is zero. Collect terms and write it as a linear combination of β_j's with coefficients in $\mathbb{Q}(\alpha)$. Conclude these coefficients in front of β_j's are zero using the independence of β_j's over $\mathbb{Q}(\alpha)$. Then use the indepence of α^i's over $\mathbb{Q}$ to conclude the oringinal coefficients are all zero.

Part (3): write the basis elements in S in the following order:

$$\{\alpha^0\beta_0, \alpha^1\beta_0, \cdots, \alpha^{m-1}\beta_0\}, \{\alpha^0\beta_1, \alpha^1\beta_1, \cdots, \alpha^{m-1}\beta_1\}, \cdots, \{\alpha^0\beta_{n-1}, \alpha^1\beta_{n-1}, \cdots, \alpha^{m-1}\beta_{n-1}\}.$$

Then your matrix for L_α will be in a block diagonal form. All blocks along the diagonal are identical $m \times m$ matrices with integer entries, most of which are zero. Hope this is sufficient information for you to figure out what this matrix is.

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