EXTRA HINTS FOR EXERCISE SHEET 10

Exercise 10.1. Part (1) is similar to Example 10.9. Part (2) is similar to Example 10.13, except that we can actually find a principal ideal of norm 2 (e.g. it is easy to find that the ideal $I = (2 + \sqrt{6})$ has norm 2). Since $\mathfrak{p}$ is the only ideal of norm 2 by Proposition 10.10, we can conclude that $\mathfrak{p} = (2+\sqrt{6})$ is a principal ideal. What does it tell you about the class number?

Part (3) is slightly more complicated, as you need to find all ideals of norm not larger than 4. Ideals of norm 1, 2 or 3 can be found in the standard way. Using the method in Example 10.13 you can prove that the only ideal of norm 2 is not principal. It remains to find ideals of norm 4. Since every ideal of norm 4 must be a factor (a product of some prime ideal factors) of the ideal (4), we need to factor (4) first. This can be done by $(4) = (2)(2) = \mathfrak{p}^2\mathfrak{p}^2 = \mathfrak{p}^4$. So every factor of (4) is a power of $\mathfrak{p}$ (not higher than 4th power). Since $N(\mathfrak{p}) = 2$, and the norm of ideals is completely multiplicative (Lemma 10.2), we can conclude that the only ideal of norm 4 is $\mathfrak{p}^2 = (2)$, which is a prime ideal. At this point the class number should be clear.

Exercise 10.2. This problem is another nice proof of Fermat's two square problem. (Remember we had a proof in Exercise 5.4.) The proof is broken into 5 simple steps. Part (1) follows from the fact that -1 is a quadratic residue modulo p (why?). Part (2) is straightforward by the formula for the volume of the fundamental domain. Part (3) is a direct application of Minkowski's Theorem 9.11. For part (4), you can write $(a, b) = m_1e_1 + m_2e_2$ for some $m_1, m_2 \in \mathbb{Z}$. Comparing the two coordinates on both sides, you can get the first congruence. The first congruence and part (1) together give the second congruence. Part (5) follows from parts (3) and (4).

Exercise 10.3. The proof for parts (1) to (4) is very similar to the proof of Proposition 10.1. Part (1) is elementary. Part (2) is an application of Corollary 9.12. Notice that the definition of L_I for this problem is contained in Proposition 9.14. Part (3) follows from the definition of L_I in Proposition 9.14 again. For part (4), you need to explain why $N(\alpha) = xy$ first, then use parts (1) and (3) to explain the inequality. The proof for part (5) is almost identical to the proof of Proposition 10.3 (with only minor changes in some coefficients).

Exercise 10.4. If you would like me to clarify any concepts, or show more examples of computations of any particular type, please feel free to let me know. I need ideals for the three lectures in week 11. Thanks!

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