Solutions to Exercise Sheet 7

Solution 7.1. *Examples of discriminants.*

(1) By the definition of the discriminants, we need to compute

$$\Delta(1, \sqrt[3]{2}, \sqrt[3]{4}) = \det \begin{pmatrix} T(1) & T(\sqrt[3]{2}) & T(\sqrt[3]{4}) \\ T(\sqrt[3]{2}) & T(\sqrt[3]{4}) & T(2) \\ T(\sqrt[3]{4}) & T(2) & T(2\sqrt[3]{2}) \end{pmatrix}.$$

By Exercise 6.2 (1), if $\alpha = a + b\sqrt[3]{2} + c\sqrt[3]{4} \in K$ for some $a, b, c \in \mathbb{Q}$, then the trace of α in $\mathbb{Q}(\sqrt[3]{2})$ is given by $T(\alpha) = 3a$. Hence we have $T(1) = 3$, $T(2) = 6$, while $T(\sqrt[3]{2}) = T(\sqrt[3]{4}) = T(2\sqrt[3]{2}) = 0$. Therefore

$$\Delta(1, \sqrt[3]{2}, \sqrt[3]{4}) = \det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{pmatrix} = -108.$$

(2) The discriminant that we need to compute is given by

$$\Delta(1, \zeta, \zeta^2, \zeta^3) = \det \begin{pmatrix} T(1) & T(\zeta) & T(\zeta^2) & T(\zeta^3) \\ T(\zeta) & T(\zeta^2) & T(\zeta^3) & T(\zeta^4) \\ T(\zeta^2) & T(\zeta^3) & T(\zeta^4) & T(1) \\ T(\zeta^3) & T(\zeta^4) & T(1) & T(\zeta) \end{pmatrix}.$$

Following the method in Exercise 6.2 (2), we can write down the matrices corresponding to L_1, L_ζ, L_{ζ^2}, L_{ζ^3} and L_{ζ^4} under the basis $\{1, \zeta, \zeta^2, \zeta^3\}$ to compute the corresponding traces. More precisely, we have $T(1) = 4$ by Lemma 6.16 (3) and

$$T(\zeta) = \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = -1; \qquad T(\zeta^2) = \operatorname{tr} \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} = -1;$$

$$T(\zeta^3) = \operatorname{tr} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} = -1; \qquad T(\zeta^4) = \operatorname{tr} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} = -1.$$

Therefore, the discriminant can be computed as

$$\Delta(1, \zeta, \zeta^2, \zeta^3) = \det \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 4 \\ -1 & -1 & 4 & -1 \end{pmatrix} = \det \begin{pmatrix} 5 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 5 \\ 0 & -1 & 5 & 0 \end{pmatrix} = 125.$$

Solution 7.2. *The discriminant of an ideal.* Since $\{\beta_1, \beta_2, \cdots, \beta_n\}$ is an integral basis for I, for each i we can write $\alpha_i = \sum_{j=1}^{n} a_{ij}\beta_j$, such that all entries of the transition matrix $M = (a_{ij})$ are integers. By Proposition 7.6, we get

$$\Delta(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\det M)^2 \Delta(\beta_1, \beta_2, \cdots, \beta_n). \tag{7.2}$$

Similarly we can write $\beta_i = \sum_{j=1}^{n} b_{ij}\alpha_j$ and all entries of the transition matrix $N = (b_{ij})$ are also integers. By Proposition 7.6 we also get

$$\Delta(\beta_1, \beta_2, \cdots, \beta_n) = (\det N)^2 \Delta(\alpha_1, \alpha_2, \cdots, \alpha_n). \tag{7.3}$$

By (7.2) and (7.3), we get

$$\Delta(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\det M)^2 (\det N)^2 \Delta(\alpha_1, \alpha_2, \cdots, \alpha_n).$$

Since $\Delta(\alpha_1, \alpha_2, \cdots, \alpha_n) \neq 0$ by Proposition 7.5, we get $(\det M)^2(\det N)^2 = 1$. Since all entries of M and N are integers, $(\det M)^2$ and $(\det N)^2$ are both non-negative integers, hence $(\det M)^2 = (\det N)^2 = 1$, and the statement we want to prove follows.

Solution 7.3. *The discriminant of a quadratic field.*

(1) In Example 6.18 we know that, for any $\alpha = a + b\sqrt{d} \in K$ for $a, b \in \mathbb{Q}$, its trace in K is given by $T(\alpha) = 2a$. Therefore we have

$$\Delta(1, \sqrt{d}) = \det \begin{pmatrix} T(1) & T(\sqrt{d}) \\ T(\sqrt{d}) & T(d) \end{pmatrix} = \det \begin{pmatrix} 2 & 0 \\ 0 & 2d \end{pmatrix} = 4d.$$

Since $\{1, \sqrt{d}\}$ is an integral basis by Proposition 7.2, we conclude that $\Delta_K = 4d$ for the quadratic field $K = \mathbb{Q}(\sqrt{d})$ when $d \equiv 2$ or $3 \pmod 4$.

(2) We still use the same formula as in part (1). Notice that $T(\frac{1+\sqrt{d}}{2}) = 2 \cdot \frac{1}{2} = 1$ and $T((\frac{1+\sqrt{d}}{2})^2) = T(\frac{1+d+2\sqrt{d}}{4}) = 2 \cdot \frac{1+d}{4} = \frac{1+d}{2}$. Then we have

$$\Delta\left(1, \frac{1+\sqrt{d}}{2}\right) = \det \begin{pmatrix} T(1) & T\left(\frac{1+\sqrt{d}}{2}\right) \\ T\left(\frac{1+\sqrt{d}}{2}\right) & T\left(\left(\frac{1+\sqrt{d}}{2}\right)^2\right) \end{pmatrix} = \det \begin{pmatrix} 2 & 1 \\ 1 & \frac{1+d}{2} \end{pmatrix} = d.$$

Since $\{1, \frac{1+\sqrt{d}}{2}\}$ is an integral basis by Proposition 7.2, we conclude that $\Delta_K = d$ for the quadratic field $K = \mathbb{Q}(\sqrt{d})$ when $d \equiv 1 \pmod 4$.

Solution 7.4. *Integral basis of a principal ideal.*

(1) Since $\alpha \in I$ and $\omega_i \in \mathcal{O}_K$ for each i, by the definition of an ideal, we get $\alpha\omega_i \in I$ for each i.

(2) Assume $b_1\alpha\omega_1 + b_2\alpha\omega_2 + \cdots + b_n\alpha\omega_n = 0$ for some $b_1, b_2, \cdots, b_n \in \mathbb{Q}$. Since $\alpha \neq 0$ we get $b_1\omega_1 + b_2\omega_2 + \cdots + b_n\omega_n = 0$. It follows that $b_i = 0$ for each i, since $\{\omega_1, \omega_2, \cdots, \omega_n\}$ is a $\mathbb{Q}$-basis for K. Therefore $\{\alpha\omega_1, \alpha\omega_2, \cdots, \alpha\omega_n\}$ are $\mathbb{Q}$-independent. Thus they form a $\mathbb{Q}$-basis for K.

(3) Every $\gamma \in I = (\alpha)$ can be written as $\gamma = \alpha\beta$ for some $\beta \in \mathcal{O}_K$. Since $\{\omega_1, \omega_2, \cdots, \omega_n\}$ is an integral basis for $\mathcal{O}_K$, we can write $\beta = b_1\omega_1 + b_2\omega_2 + \cdots + b_n\omega_n$ for $b_1, b_2, \cdots, b_n \in \mathbb{Z}$. Hence $\gamma = b_1\alpha\omega_1 + b_2\alpha\omega_2 + \cdots + b_n\alpha\omega_n$ is an integral linear combination of $\{\alpha\omega_1, \alpha\omega_2, \cdots, \alpha\omega_n\}$. Together with the result in part (2), we conclude that $\alpha\omega_1, \alpha\omega_2, \cdots, \alpha\omega_n$ is an integral basis for I.

(4) By Proposition 7.2, an integral basis for $\mathcal{O}_K$ is given by $\{\omega_1 = 1, \omega_2 = \sqrt{3}\}$. By the conclusion in part (3), an integral basis for I is given by $\{\alpha\omega_1 = \sqrt{3}, \alpha\omega_2 = 3\}$. Therefore we have

$$\Delta(I) = \Delta(\sqrt{3}, 3) = \det\begin{pmatrix} T(3) & T(3\sqrt{3}) \\ T(3\sqrt{3}) & T(9) \end{pmatrix} = \det\begin{pmatrix} 6 & 0 \\ 0 & 18 \end{pmatrix} = 108.$$