Solutions to Exercise Sheet 9

Solution 9.1. *Card games and non-card games.*

(1) A diamond is convex assuming the four sides are all line segments (despite that they look a little curved on any playing cards). All the other shapes are non-convex.

(2) The shapes (ii), (iv) and (vi) are centrally symmetric. The other shapes are not.

Solution 9.2. *Applications of Minkowski's Theorem.*

(1) D is centrally symmetric, convex and compact. Hence Corollary 9.12 applies. If $\mathrm{vol}(D) \geqslant 4A$, then D is guaranteed to contain a non-zero point in L. This condition can be written as $\pi r^2 \geqslant 4A$. When $r > 0$, it is equivalent to $r \geqslant \left(\frac{4A}{\pi}\right)^{\frac{1}{2}}$.

(2) S is centrally symmetric, convex and compact. Hence Corollary 9.12 applies. If $\mathrm{vol}(S) \geqslant 4A$, then S is guaranteed to contain a non-zero point in L. Note that $\mathrm{vol}(S) = 2r^2$, hence this condition becomes $2r^2 \geqslant 4A$. When $r > 0$, it is equivalent to $r \geqslant (2A)^{\frac{1}{2}}$.

Solution 9.3. *Basic properties of ideal classes.*

(1) The reflexivity is clear, as for any non-zero principal ideal (α), we have $(\alpha)I = (\alpha)I$, hence $I \sim I$. The symmetry is also easy. If $I \sim J$, then there exist non-zero principal ideals (α) and (β), such that $(\alpha)I = (\beta)J$. We switch the two sides and write the equation as $(\beta)J = (\alpha)I$, then by definition we get $J \sim I$.

Now we prove the transitivity. By $I_1 \sim I_2$, we can find non-zero principal ideals (α_1) and (α_2), such that $(\alpha_1)I_1 = (\alpha_2)I_2$. By $I_2 \sim I_3$, we can find non-zero principal ideals (β_2) and (β_3), such that $(\beta_2)I_2 = (\beta_3)I_3$. We multiply both sides of the first identity by (β_2) and get $(\alpha_1)(\beta_2)I_1 = (\alpha_2)(\beta_2)I_2$. By Exercise 8.2 (3), we can rewrite it as $(\alpha_1\beta_2)I_1 = (\alpha_2\beta_2)I_2$. Similarly, we can multiply both sides of the second identity by (α_2) to get $(\alpha_2)(\beta_2)I_2 = (\alpha_2)(\beta_3)I_3$, which can be rewritten as $(\alpha_2\beta_2)I_2 = (\alpha_2\beta_3)I_3$. Now we get $(\alpha_1\beta_2)I_1 = (\alpha_2\beta_2)I_2 = (\alpha_2\beta_3)I_3$. We need to show that $(\alpha_1\beta_2)$ and $(\alpha_2\beta_3)$ are both non-zero principal ideals. Since α_1 and β_2 are both non-zero complex numbers, their product $\alpha_1\beta_2$ is also non-zero, hence $(\alpha_1\beta_2)$ is also a non-zero principal ideal. For the same reason $(\alpha_2\beta_3)$ is a non-zero principal ideal. Hence we conclude that $I_1 \sim I_3$.

(2) From $I_1 \sim I_2$, we know that for some non-zero principal ideals (α_1) and (α_2), we have $(\alpha_1)I_1 = (\alpha_2)I_2$. $J_1 \sim J_2$, we know that for some non-zero principal ideals (β_1) and (β_2), we have $(\beta_1)J_1 = (\beta_2)J_2$. We multiply the two identities to get $(\alpha_1)(\beta_1)I_1J_1 = (\alpha_2)(\beta_2)I_2J_2$. By Exercise 8.2 (3), we can rewrite it as

$(\alpha_1\beta_1)I_1J_1 = (\alpha_2\beta_2)I_2J_2$. For similar reasons as in part (1), both $(\alpha_1\beta_1)$ and $(\alpha_2\beta_2)$ are non-zero principal ideals. Hence we have $I_1J_1 \sim I_2J_2$.

Solution 9.4. *Volume of the fundamental domain for real quadratic fields.*

(1) We prove that L_I is a lattice of rank 2 in $\mathbb{R}^2$. By Proposition 7.9, assume α_1, α_2 is an integral basis for I, then we can write $I = \{m_1\alpha_1 + m_2\alpha_2 \mid m_1, m_2 \in \mathbb{Z}\}$. We write $\alpha_1 = a_1 + b_1\sqrt{d}$ and $\alpha_2 = a_2 + b_2\sqrt{d}$ for some $a_1, b_1, a_2, b_2 \in \mathbb{Q}$. Let $e_1 = (a_1 + b_1\sqrt{d}, a_1 - b_1\sqrt{d})$ and $e_2 = (a_2 + b_2\sqrt{d}, a_2 - b_2\sqrt{d})$, then for every $\alpha = m_1\alpha_1 + m_2\alpha_2 = (m_1a_1 + m_2a_2) + (m_1b_1 + m_2b_2)\sqrt{d} \in I$, the corresponding point in L_I is given by $((m_1a_1+m_2a_2)+(m_1b_1+m_2b_2)\sqrt{d}, (m_1a_1+m_2a_2)-(m_1b_1+m_2b_2)\sqrt{d}) = m_1(a_1+b_1\sqrt{d}, a_1-b_1\sqrt{d})+m_2(a_2+b_2\sqrt{d}, a_2-b_2\sqrt{d}) = m_1e_1+m_2e_2$. Hence $L_I = \{m_1e_1 + m_2e_2 \mid m_1, m_2 \in \mathbb{Z}\}$ is a rank 2 lattice in $\mathbb{R}^2$.

(2) We calculate $T_{\mathcal{O}_K}$. By Proposition 7.2, we can write $\mathcal{O}_K = \{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}\}$, where $\omega_1 = 1$, and $\omega_2 = \sqrt{d}$ if $d \equiv 2$ or $3 \pmod 4$ and $\frac{1}{2}(1 + \sqrt{d})$ if $d \equiv 1 \pmod 4$.

When $d \equiv 2$ or $3 \pmod 4$, we have $e_1 = (1, 1)$ and $e_2 = (\sqrt{d}, -\sqrt{d})$. Hence

$$\mathrm{vol}(T_{\mathcal{O}_K}) = \left|\det\begin{pmatrix}1 & \sqrt{d}\\ 1 & -\sqrt{d}\end{pmatrix}\right| = \left|-2\sqrt{d}\right| = 2\sqrt{d} = |\Delta_K|^{\frac{1}{2}},$$

where the last equality follows from Proposition 7.14.

When $d \equiv 1 \pmod 4$, we have $e_1 = (1, 1)$ and $e_2 = (\frac{1}{2}(1 + \sqrt{d}), \frac{1}{2}(1 - \sqrt{d}))$. Hence the volume of the fundamental domain is

$$\mathrm{vol}(T_{\mathcal{O}_K}) = \left|\det\begin{pmatrix}1 & \frac{1}{2}(1+\sqrt{d})\\ 1 & \frac{1}{2}(1-\sqrt{d})\end{pmatrix}\right| = \left|-\sqrt{d}\right| = \sqrt{d} = |\Delta_K|^{\frac{1}{2}},$$

where the last equality still follows from Proposition 7.14.

(3) We calculate the volume of the fundamental domain T_I in general. For an arbitrary ideal I with an integral basis α_1, α_2, we can write $\alpha_1 = a_{11}\omega_1 + a_{21}\omega_2$ and $\alpha_2 = a_{12}\omega_1 + a_{22}\omega_2$, as well as the transition matrix $M = (a_{ij})$, where $a_{ij} \in \mathbb{Z}$. For simplicity, we write the points in L_I corresponding to α_i by (α_i, α_i') for $i = 1, 2$. Similarly, we write the points in L_I corresponding to ω_i by (ω_i, ω_i') for $i = 1, 2$. Then they can be organised into the following matrix

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_1' & \alpha_2'\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\omega_1 & \omega_2\\ \omega_1' & \omega_2'\end{pmatrix}.$$

Taking determinants and absolute values on both sides, we get

$$\mathrm{vol}(T_I) = |\det M|\,\mathrm{vol}(T_{\mathcal{O}_K}).$$

By Proposition 8.3 and part (2), we conclude that

$$\mathrm{vol}(T_I) = N(I)\,|\Delta_K|^{\frac{1}{2}}.$$