Math 19 Calculus **Winter 2007** **Instructor: Geir Helleloid**

Final Exam Solutions

1. (a) (10 points) Let

$$y = \frac{2}{x-1} - \frac{1}{\sqrt{x}}.$$

Find dy/dx.

Solution.

$$\begin{aligned} y &= 2(x-1)^{-1} - x^{-1/2} \\ \frac{dy}{dx} &= -2(x-1)^{-2} + \frac{1}{2}\,x^{-3/2} \\ &= \boxed{-\frac{2}{(x-1)^2} + \frac{1}{2x^{3/2}}} \end{aligned}$$

□

(b) (10 points) Let

$$y = (\sin x)^{\cos x}.$$

Find dy/dx. Your answer should be a function of x only.

Solution.

$$\begin{aligned} y &= (\sin x)^{\cos x} \\ \ln y &= \cos x \ln(\sin x) \\ \frac{1}{y}\cdot\frac{dy}{dx} &= -\sin x \ln(\sin x) + \cos x \cdot \frac{\cos x}{\sin x} \\ \frac{dy}{dx} &= y\left(-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right) \\ &= \boxed{(\sin x)^{\cos x}\left(-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right)} \end{aligned}$$

□

(c) (10 points) Let

$$y = \sqrt{\tan(x^2)}.$$

Find dy/dx.

Solution.

$$\begin{aligned} y &= \left[\tan\left(x^2\right)\right]^{1/2} \\ \frac{dy}{dx} &= \frac{1}{2}\left[\tan\left(x^2\right)\right]^{-1/2} \cdot \sec^2\left(x^2\right) \cdot (2x) \\ &= \boxed{\frac{x}{\cos^2\left(x^2\right)\sqrt{\tan\left(x^2\right)}}} \end{aligned}$$

□

(d) (10 points) Find the equation of the tangent line to the curve

$$e^{x^2} + e^{y^2} = 2e$$

at the point $(-1, 1)$.

Solution.

$$\begin{aligned} e^{x^2} + e^{y^2} &= 2e \\ 2xe^{x^2} + 2ye^{y^2} \cdot \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{xe^{x^2}}{ye^{y^2}} \\ \left.\frac{dy}{dx}\right|_{(-1,1)} &= 1 \end{aligned}$$

So the equation of the tangent line is $y - 1 = 1(x + 1)$, or $\boxed{y = x + 2}$. □

(e) (10 points) Let

$$y = \frac{(2x+1)^4 \sin x}{(\ln x)\sqrt{3x-1}}.$$

Find $\frac{dy}{dx}$. Your answer should be a function of x only.

Solution.

$$\begin{aligned} y &= \frac{(2x+1)^4 \sin x}{(\ln x)\sqrt{3x-1}} \\ \ln y &= 4\ln(2x+1) + \ln(\sin x) - \ln(\ln x) - \frac{1}{2}\ln(3x-1) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{8}{2x+1} + \frac{\cos x}{\sin x} - \frac{1/x}{\ln x} - \frac{3/2}{3x-1} \\ \frac{dy}{dx} &= \boxed{\frac{(2x+1)^4 \sin x}{(\ln x)\sqrt{3x-1}}\left(\frac{8}{2x+1} + \cot x - \frac{1}{x\ln x} - \frac{3}{6x-2}\right)} \end{aligned}$$

□

2. (20 points) Let

$$f(x) = \ln(x^2 - 1).$$

(a) (10 points) You must show all your work, but please write your final answers in the box.

The domain of $f(x)$ is:	$(-\infty, -1) \cup (1, \infty)$
$f(x)$ is increasing on:	$(1, \infty)$
$f(x)$ is decreasing on:	$(-\infty, -1)$
$f(x)$ has local maxima at:	None
$f(x)$ has local minima at:	None
$f(x)$ is concave up on:	None
$f(x)$ is concave down on:	$(-\infty, -1) \cup (1, \infty)$

Solution.

$$\begin{aligned} f(x) &= \ln(x^2 - 1) \\ f'(x) &= \frac{2x}{x^2 - 1} \\ f''(x) &= \frac{2(x^2 - 1) - (2x)(2x)}{(x^2 - 1)^2} \\ &= \frac{-2x^2 - 2}{(x^2 - 1)^2} \\ &= -\frac{2(x^2 + 1)}{(x^2 - 1)^2} \end{aligned}$$

Then $f'(x) > 0$ for $x > 1$ and $f'(x) < 0$ for $x < -1$. Also, $f''(x) > 0$ for $x < -1$ and for $x > 1$. □

(b) (4 points) Compute the following four limits.

Solution.

$$\lim_{x \to \infty} \ln(x^2 - 1) = \infty$$

$$\lim_{x \to -\infty} \ln(x^2 - 1) = \infty$$

$$\lim_{x \to 1^+} \ln(x^2 - 1) = -\infty$$

$$\lim_{x \to -1^-} \ln(x^2 - 1) = -\infty$$

□

(c) (1 point) List all vertical and horizontal asymptotes of $y = \ln(x^2 - 1)$.

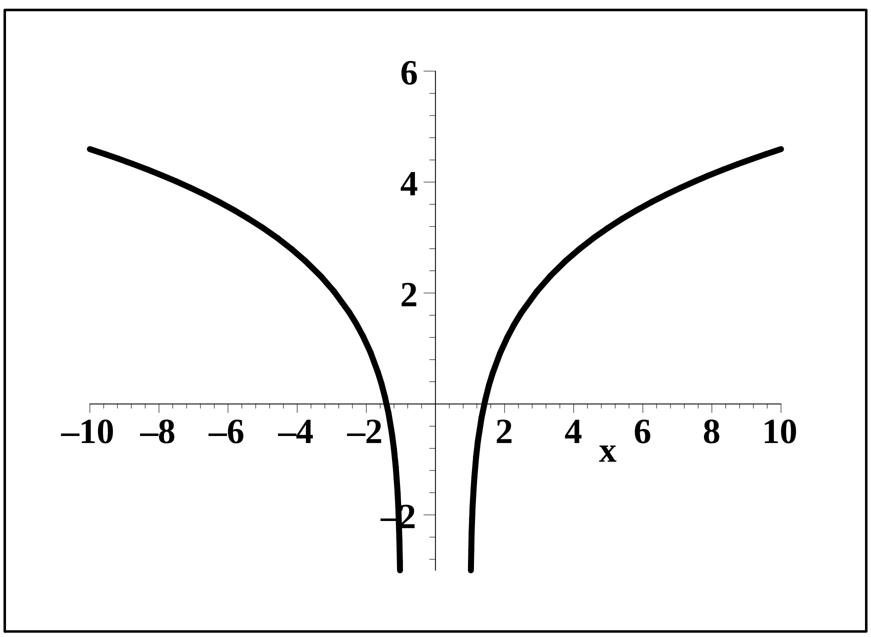
Solution. There are no horizontal asymptotes and there are two vertical asymptotes at $x = \pm 1$. □

(d) (5 points) Using your answers from parts (a) and (b), sketch a graph of

$$f(x) = \ln(x^2 - 1).$$

Even if your answers in parts (a) and (b) are wrong, if your sketch correctly uses those answers, you may earn partial credit.

Solution.



□

3. (20 points) A particle is moving along the curve $x^2 - 4xy - y^2 = -5$. Given that the x-coordinate of the particle is changing at 3 units/second, *how fast is the distance from the particle to the origin changing* when the particle is at the point $(1, 2)$? Hint: As an intermediate step, you should compute

$$\left.\frac{dy}{dt}\right|_{x=1,y=2}.$$

Solution. Let $y(t)$ be the y-coordinate of the particle at time t, let $x(t)$ be the x-coordinate of the particle at time t, and let $d(t)$ be the distance from the particle to the origin at time t. Then

$$d^2 = x^2 + y^2,$$

we know that $\frac{dx}{dt} = 3$, and we want to compute

$$\left.\frac{dd}{dt}\right|_{x=1,y=2}.$$

Differentiating the given equation shows

$$\begin{aligned} x^2 - 4xy - y^2 &= -5 \\ 2x \cdot \frac{dx}{dt} - 4\left(x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}\right) - 2y \cdot \frac{dy}{dt} &= 0 \\ (2x - 4y) \cdot \frac{dx}{dt} - (4x + 2y) \cdot \frac{dy}{dt} &= 0 \end{aligned}$$

When $x = 1$ and $y = 2$, we find

$$\begin{aligned} -6 \cdot 3 - 8 \cdot \frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -\frac{9}{4} \end{aligned}$$

Next, we find that

$$\begin{aligned} 2d \cdot \frac{dd}{dt} &= 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} \\ &= 6x - \frac{9}{2} \cdot y \end{aligned}$$

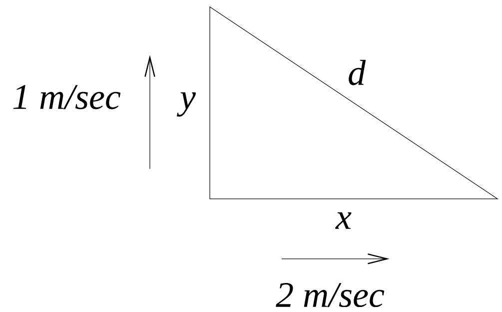
When $x = 1$ and $y = 2$, we know $d = \sqrt{5}$ and

$$\begin{aligned} 2\sqrt{5} \cdot \frac{dd}{dt} &= -3 \\ \frac{dd}{dt} &= \boxed{-\frac{3}{2\sqrt{5}}} \end{aligned}$$

□

4. (20 points) A balloon is rising at a constant speed of 1 m/sec. A girl is cycling along a straight road at a speed of 2 m/sec. When she passes under the balloon it is 3 m above her. How fast is the distance between the girl and the balloon increasing 2 seconds later?

Solution.



Let x be the distance the girl has traveled, let y be the altitude of the balloon, and let d be the distance between them. Then

$$\begin{aligned} d^2 &= x^2 + y^2 \\ 2d\,\frac{dd}{dt} &= 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \\ 2d\,\frac{dd}{dt} &= 4x + 2y \end{aligned}$$

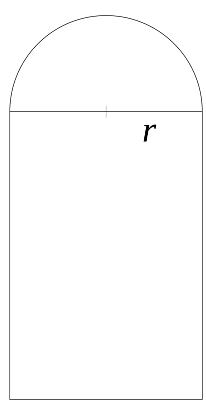
After 2 seconds, we know that $x = 4$, $y = 5$, and $z = \sqrt{41}$. So

$$\begin{aligned} 2\sqrt{41}\cdot\frac{dd}{dt} &= 26 \\ \frac{dd}{dt} &= \boxed{\frac{13}{\sqrt{41}}} \end{aligned}$$

After 2 seconds, the distance between the girl and the balloon is increasing at $\frac{13}{\sqrt{41}}$ m/sec. □

5. (20 points) A Norman window consists of a rectangle surmounted by a semicircle, as shown. Given that the total area of the window is $A = 8 + 2\pi$, find the minimum possible perimeter of the window. (Please note the horizontal line between the rectangle and the semicircle does not count as part of the perimeter.) Hint: The total area has been carefully chosen so that the minimum perimeter occurs at a very simple value of r. If your optimal value of r is complicated, you have done something incorrectly.

Solution.



Let A be the area and P be the perimeter. Then

$$\begin{aligned} A &= \frac{1}{2}\pi r^2 + 2rh \\ P &= \pi r + 2h + 2r \end{aligned}$$

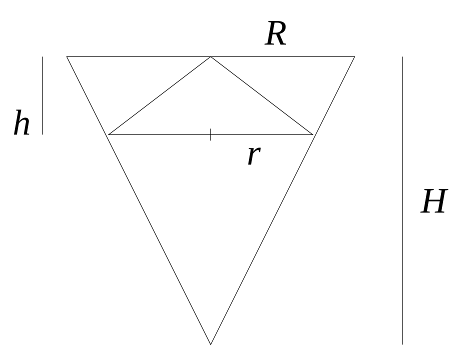
Therefore

$$\begin{aligned} 8 + 2\pi &= \frac{1}{2}\pi r^2 + 2rh \\ 8 + 2\pi - \frac{1}{2}\pi r^2 &= 2rh \\ h &= \frac{4}{r} + \frac{\pi}{r} - \frac{1}{4}\pi r \\ P &= \pi r + \frac{8}{r} + \frac{2\pi}{r} - \frac{1}{2}\pi r + 2r \\ &= \left(2 + \frac{\pi}{2}\right) r + \frac{8 + 2\pi}{r} \\ P'(r) &= 2 + \frac{\pi}{2} - \frac{8 + 2\pi}{r^2} \\ 0 &= \frac{4 + \pi}{2} - \frac{8 + 2\pi}{r^2} \\ r^2 &= 4 \\ r &= \pm 2 \end{aligned}$$

The domain of $P(r)$ is $(0, \infty)$. For $0 < r < 2$, $P'(r)$ is negative, and for $r > 2$, $P'(r)$ is positive. Therefore, the critical point $r = 2$ is an absolute minimum. When $r = 2$, the perimeter is $\boxed{8 + 2\pi}$, and this is the minimum perimeter. □

6. (20 points) Suppose you have a cone with *constant* height H and *constant* radius R, and you want to put a smaller cone "upside down" inside the larger cone (see the picture). If h is the height of the smaller cone, what should h be to maximize the volume of the smaller cone? The optimal value of h will depend on H. Recall that the volume of a cone with base radius r and height h is given by the formula $V = \frac{1}{3}\pi r^2 h$.

Solution.



Using similar triangles, we find that

$$\frac{r}{R} = \frac{H-h}{H},$$

or

$$r = \frac{R(H-h)}{H}.$$

Then, if V is the volume of the smaller cone,

$$\begin{aligned}
V &= \frac{1}{3}\,\pi r^2 h \\
&= \frac{1}{3}\,\pi\left(\frac{R(H-h)}{H}\right)^2 h \\
&= \frac{\pi R^2}{3H^2}\,h(H-h)^2 \\
&= \frac{\pi R^2}{3H^2}\,(H^2 h - 2Hh^2 + h^3) \\
V'(h) &= \frac{\pi R^2}{3H^2}\,(H^2 - 4Hh + 3h^2) \\
&= \frac{\pi R^2}{3H^2}\,(H-3h)(H-h)
\end{aligned}$$

So the critical points of $V(h)$ are $h = H$ and $h = \frac{H}{3}$. The domain of $V(h)$ is $[0, H]$. By the Closed Interval Method, the maximum value of $V(h)$ occurs at $h = 0$, $h = \frac{H}{3}$, or $h = H$. Since $V(0) = V(H) = 0$, it follows that the volume is maximized at $\boxed{h = H}$. □

7. (10 points) For parts (a) and (b), compute the given limits, if they exist. If you assert that a limit does not exist, you need to justify your answer to get full credit.

   (a) (5 points)

$$\lim_{x\to\infty}\left(\sqrt{x^2-3x+1} - \sqrt{x^2+2}\right)$$

Solution.

$$\begin{aligned}
&\lim_{x\to\infty}\left(\sqrt{x^2-3x+1}-\sqrt{x^2+2}\right)\\
=\ &\lim_{x\to\infty}\frac{\sqrt{x^2-3x+1}-\sqrt{x^2+2}}{1}\cdot\frac{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}\\
=\ &\lim_{x\to\infty}\frac{(x^2-3x+1)-(x^2+2)}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}\\
=\ &\lim_{x\to\infty}\frac{-3x-1}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}\\
=\ &\lim_{x\to\infty}\frac{-3x-1}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}\cdot\frac{1/x}{1/x}\\
=\ &\lim_{x\to\infty}\frac{-3-\frac{1}{x}}{\sqrt{1-\frac{3}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{2}{x^2}}}\\
=\ &\boxed{-\frac{3}{2}}
\end{aligned}$$

□

(b) (5 points)

$$\lim_{x\to 2} e^{\frac{1}{x-2}}$$

Solution.

$$\begin{aligned}
\lim_{x\to 2^+} e^{\frac{1}{x-2}} &= e^{\frac{1}{0^+}} = e^{\infty} = \infty\\
\lim_{x\to 2^-} e^{\frac{1}{x-2}} &= e^{\frac{1}{0^-}} = e^{-\infty} = 0
\end{aligned}$$

Since the right- and left-hand limits differ, the limit $\boxed{\text{does not exist}}$. □