Final Exam
Solutions

Name: ______________________________

I agree to abide by the honor code:

Signature: ______________________________

- You have 3 hours.
- No notes, books, or calculators are permitted.
- **You must show all work to receive credit!**
- Please check your solutions carefully.

1. ________ (/50 points)
2. ________ (/20 points)
3. ________ (/30 points)
4. ________ (/20 points)
5. ________ (/20 points)
6. ________ (/20 points)
7. ________ (/20 points)
8. ________ (/10 points)
9. ________ (/5 points)

Total. ________ (/195 points)

1. (a) (10 points) Let

$$y = \frac{2}{x-1} - \frac{x+2}{\sqrt{x}}.$$

Find dy/dx.

Solution.

$$\begin{aligned}
y &= 2(x-1)^{-1} - \frac{x+2}{\sqrt{x}} \\
\frac{dy}{dx} &= -2(x-1)^{-2} - \frac{\sqrt{x} - (x+2)\frac{1}{2}x^{-\frac{1}{2}}}{x} \\
&= -2(x-1)^{-2} - \frac{x - \frac{1}{2}(x+2)}{x^{\frac{3}{2}}} \\
&= \frac{-2}{(x-1)^2} - \frac{\frac{1}{2}x + 1}{x^{\frac{3}{2}}}
\end{aligned}$$

□

(b) (10 points) Let

$$y = (\sin 2x)^x.$$

Find dy/dx. Your answer should be a function of x only.

Solution.

$$\begin{aligned}
y &= (\sin 2x)^x \\
\ln y &= x \ln(\sin 2x) \\
\frac{1}{y} \cdot \frac{dy}{dx} &= \ln(\sin 2x) + x \cdot \frac{2\cos 2x}{\sin 2x} \\
\frac{dy}{dx} &= y\left(\ln(\sin 2x) + x \cdot \frac{2\cos 2x}{\sin 2x}\right) \\
\frac{dy}{dx} &= (\sin 2x)^x \left(\ln(\sin 2x) + x \cdot \frac{2\cos 2x}{\sin 2x}\right)
\end{aligned}$$

□

(c) (10 points) Let

$$y = \ln(\tan(x^2)).$$

Find dy/dx.

Solution.

$$\begin{aligned} y &= \ln(\tan(x^2)) \\ \frac{dy}{dx} &= \frac{1}{\tan(x^2)} \cdot \sec^2(x^2) \cdot (2x) \\ &= \cot(x^2) \cdot \sec^2(x^2) \cdot (2x) \end{aligned}$$

We could simplify this in various ways; $\frac{dy}{dx} = \frac{2x}{\sin(x^2)\cos(x^2)}$ or $\frac{dy}{dx} = \frac{4x}{\sin(2x^2)}$ are examples. □

(d) (10 points) Find the equation of the tangent line to the curve

$$e^{x^2} + e^{y^2} = 2e$$

at the point $(-1, 1)$.

Solution.

$$\begin{aligned} e^{x^2} + e^{y^2} &= 2e \\ 2xe^{x^2} + 2ye^{y^2} \cdot \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{xe^{x^2}}{ye^{y^2}} \\ \left.\frac{dy}{dx}\right|_{(-1,1)} &= 1 \end{aligned}$$

So the equation of the tangent line is $y - 1 = 1(x + 1)$, or $y = x + 2$. □

(e) (10 points) Let

$$y = \frac{(2x+1)^4 \cos(x^2)}{(\ln x)\sqrt{3x-1}}.$$

Find $\frac{dy}{dx}$. Your answer should be a function of x only.

Solution.

$$\begin{aligned}
y &= \frac{(2x+1)^4 \sin x}{(\ln x)\sqrt{3x-1}} \\
\ln y &= 4\ln(2x+1) + \ln(\sin x) - \ln(\ln x) - \frac{1}{2}\ln(3x-1) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{8}{2x+1} + \frac{\cos x}{\sin x} - \frac{1/x}{\ln x} - \frac{3/2}{3x-1} \\
\frac{dy}{dx} &= \frac{(2x+1)^4 \sin x}{(\ln x)\sqrt{3x-1}}\left(\frac{8}{2x+1} + \cot x - \frac{1}{x\ln x} - \frac{3}{6x-2}\right)
\end{aligned}$$

□

2. (20 points) Let

$$f(x) = x^3 - 3x + 1.$$

(a) (10 points) You must show all your work, but please write your final answers in the box.

The domain of $f(x)$ is:	$(-\infty, \infty)$
$f(x)$ is increasing on:	$(-\infty, -1) \cup (1, \infty)$
$f(x)$ is decreasing on:	$(-1, 1)$
$f(x)$ has local maxima at:	$x = -1$
$f(x)$ has local minima at:	$x = 1$
$f(x)$ is concave up on:	$(0, \infty)$
$f(x)$ is concave down on:	$(-\infty, 0)$

Solution. Since f is a polynomial, the domain of f is all real numbers.

$$\begin{aligned} f(x) &= x^3 - 3x + 1 \\ f'(x) &= 3x^2 - 3 \\ f'(x) &= 3(x^2 - 1) = 3(x+1)(x-1) \end{aligned}$$

The critical points of f are therefore $x = 1$ and $x = -1$. We see that f' is positive on the union of intervals $(-\infty, -1) \cup (1, \infty)$ and negative on the interval $(-1, 1)$. Therefore, f is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$.
Since the first derivative switches sign from positive to negative at $x = -1$, f has a local maximum there. Since f' switches from negative to positive at $x = 1$, f has a local minimum there.

$$\begin{aligned} f'(x) &= 3(x^2 - 1) \\ f''(x) &= 3(2x) = 6x \end{aligned}$$

f'' is positive on $(0, \infty)$ and negative on $(-\infty, 0)$. Therefore, f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$ with a point of inflection at $x = 0$ (where there is a change in concavity). We can also note that $f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$ which is consistent with there being a maximum at $x = -1$ and a minimum at $x = 1$ (since $f'(-1) = f'(1) = 0$). □

(b) (5 points) Give the number of zeros (roots) of $f(x) = x^3 - 3x + 1$. Justify your answer. (Hint: One possible solution involves computing $f(-2), f(-1), f(1), f(2)$ and using that f is a continuous function.)

Solution. Since f is a continuous function, the intermediate value theorem says that if f changes sign on an interval, f must have a zero on that interval.

$$\begin{aligned} f(-2) &= -8 + 6 + 1 = -1 \\ f(-1) &= -1 + 3 + 1 = 3 \\ f(1) &= 1 - 3 + 1 = -1 \\ f(2) &= 8 - 6 + 1 = 3 \end{aligned}$$

Therefore f has zeros on the intervals $(-2, -1), (-1, 1)$, and $(1, 2)$. Since f is a polynomial of degree three, it factors as the product of those roots and so has exactly three roots. □

(c) (5 points) Using all of the above, sketch a graph of

$$f(x) = x^3 - 3x + 1.$$

Even if your answers in parts (a) and (b) are wrong, if your sketch correctly uses those answers, you may earn partial credit.

Solution. It is useful to know that

$$\lim_{x \to -\infty} = -\infty$$

and

$$\lim_{x \to \infty} = \infty.$$

One way to see this is to use the fact that

$$\lim_{x \to \pm\infty} \frac{f(x)}{x^3} = 1.$$

We also note that we know the value of the function at its local extrema from our calculations in (b). Using the rest of the data from (a), we can make the following sketch: □

3. (30 points) Let

$$f(x) = \frac{x^2+1}{x^2-1}.$$

(a) (10 points) You must show all your work, but please write your final answers in the box.

The domain of $f(x)$ is:	$(-\infty,-1)\cup(-1,1)\cup(1,\infty)$
$f(x)$ is increasing on:	$(-\infty,-1)\cup(-1,0)$
$f(x)$ is decreasing on:	$(0,1)\cup(1,\infty)$
$f(x)$ has local maxima at:	$x=0$
$f(x)$ has local minima at:	there are no local minima
$f(x)$ is concave up on:	$(\infty,-1)\cup(1,\infty)$
$f(x)$ is concave down on:	$(-1,1)$

Solution. Since f is a rational function, the domain of f is all real numbers where the denominator is nonzero. Since the denominator $x^2-1=(x+1)(x-1)$, we see that the domain of f is given by $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

$$\begin{aligned} f(x) &= \frac{x^2+1}{x^2-1} \\ f'(x) &= \frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2} \\ f'(x) &= \frac{(2x)(-2)}{(x^2-1)^2} \\ f'(x) &= \frac{-4x}{(x^2-1)^2} \end{aligned}$$

f' is zero or undefined at $x=-1$, $x=0$, and $x=-1$. Since -1 and 1 are not in the domain of f, the only critical number is $x=0$.

Since the denominator of f' is positive whenever f is defined, we see that f' is positive on $(-\infty,-1)\cup(-1,0)$ and negative on $(0,1)\cup(1,\infty)$, which indicates where f is increasing and decreasing, respectively.

Since the first derivative switches sign from positive to negative at $x=0$, f has a local maximum there. There are no other critical numbers, and since local extrema occur only at critical numbers, there are no local minima.

$$
\begin{aligned}
f'(x) &= \frac{-4x}{(x^2-1)^2} \\
f''(x) &= \frac{(x^2-1)^2(-4)-(-4x)2(x^2-1)(2x)}{(x^2-1)^4} \\
f''(x) &= \frac{(x^2-1)(-4)+16x^2}{(x^2-1)^3} \\
f''(x) &= \frac{12x^2+4}{(x^2-1)^3} \\
f''(x) &= \frac{4(3x^2+1)}{(x^2-1)^3}
\end{aligned}
$$

The numerator of f'' is always positive. The denominator of f'' has zeros at $x = -1$ and $x = 1$. We can therefore use the intermediate value theorem for a number line test to see that f'' is positive on $(\infty, -1) \cup (1, \infty)$ and negative on $(-1, 1)$. That $f''(0) < 0$ and $f'(0) = 0$ is consistent with our earlier conclusion that f has a local maximum there. $\square$

(b) (15 points) Compute the following six limits.

i. $\lim_{x\to\infty} \frac{x^2+1}{x^2-1}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+1}{x^2-1} &= \lim_{x\to\infty} \frac{x^2+1}{x^2-1} \cdot \frac{x^{-2}}{x^{-2}} \\
&= \lim_{x\to\infty} \frac{1+x^{-2}}{1-x^{-2}} \\
&= \frac{\lim_{x\to\infty} 1+x^{-2}}{\lim_{x\to\infty} 1-x^{-2}} \\
&= \frac{1}{1} \\
&= 1
\end{aligned}$$

□

ii. $\lim_{x\to-\infty} \frac{x^2+1}{x^2-1}$

Solution.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^2+1}{x^2-1} &= \lim_{x\to-\infty} \frac{x^2+1}{x^2-1} \cdot \frac{x^{-2}}{x^{-2}} \\
&= \lim_{x\to-\infty} \frac{1+x^{-2}}{1-x^{-2}} \\
&= \frac{\lim_{x\to-\infty} 1+x^{-2}}{\lim_{x\to-\infty} 1-x^{-2}} \\
&= \frac{1}{1} \\
&= 1
\end{aligned}$$

□

iii. $\lim_{x\to1^-} \frac{x^2+1}{x^2-1}$

Solution. The numerator approaches 2. The denominator approaches 0 and is negative. Therefore,

$$\lim_{x\to1^-} \frac{x^2+1}{x^2-1} = -\infty$$

□

iv. $\lim_{x \to 1^+} \frac{x^2+1}{x^2-1}$

Solution. The numerator approaches 2. The denominator approaches 0 and is positive. Therefore,

$$\lim_{x \to 1^+} \frac{x^2+1}{x^2-1} = \infty$$

□

v. $\lim_{x \to -1^-} \frac{x^2+1}{x^2-1}$

Solution. The numerator approaches 2. The denominator approaches 0 and is positive. Therefore,

$$\lim_{x \to -1^-} \frac{x^2+1}{x^2-1} = \infty$$

□

vi. $\lim_{x \to -1^+} \frac{x^2+1}{x^2-1}$

Solution. The numerator approaches 2. The denominator approaches 0 and is negative. Therefore,

$$\lim_{x \to -1^+} \frac{x^2+1}{x^2-1} = -\infty$$

□

(c) (5 points) Sketch a graph of

$$f(x) = \frac{x^2 + 1}{x^2 - 1}.$$

Even if your answers in parts (a) and (b) are wrong, if your sketch correctly uses those answers, you may earn partial credit.

Solution. □

4. (20 points) A particle is moving along the curve $x^2 - 4xy - y^2 = -5$. Given that the x-coordinate of the particle is changing at 3 units/second, *how fast is the distance from the particle to the origin changing* when the particle is at the point $(1, 2)$? Hint: As an intermediate step, you should compute the value of $\frac{dy}{dt}$ when $x = 1$ and $y = 2$.

Solution. Let $y(t)$ be the y-coordinate of the particle at time t, let $x(t)$ be the x-coordinate of the particle at time t, and let $d(t)$ be the distance from the particle to the origin at time t. Then

$$d^2 = x^2 + y^2,$$

we know that $\frac{dx}{dt} = 3$, and we want to compute

$$\left.\frac{dd}{dt}\right|_{x=1,y=2}.$$

Differentiating the given equation shows

$$\begin{aligned} x^2 - 4xy - y^2 &= -5 \\ 2x \cdot \frac{dx}{dt} - 4\left(x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}\right) - 2y \cdot \frac{dy}{dt} &= 0 \\ (2x - 4y) \cdot \frac{dx}{dt} - (4x + 2y) \cdot \frac{dy}{dt} &= 0 \end{aligned}$$

When $x = 1$ and $y = 2$, we find

$$\begin{aligned} -6 \cdot 3 - 8 \cdot \frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -\frac{9}{4} \end{aligned}$$

Next, we find that

$$\begin{aligned} 2d \cdot \frac{dd}{dt} &= 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} \\ &= 6x - \frac{9}{2} \cdot y \end{aligned}$$

When $x = 1$ and $y = 2$, we know $d = \sqrt{5}$ and

$$\begin{aligned} 2\sqrt{5} \cdot \frac{dd}{dt} &= -3 \\ \frac{dd}{dt} &= -\frac{3}{2\sqrt{5}} \end{aligned}$$

□

5. (20 points) A balloon is rising at a constant speed of 1 m/sec. A girl is cycling along a straight road at a speed of 2 m/sec. When she passes under the balloon it is 3 m above her. How fast is the distance between the girl and the balloon increasing 2 seconds later?

Solution. Let x be the distance the girl has traveled, let y be the altitude of the balloon, and let d be the distance between them. Then

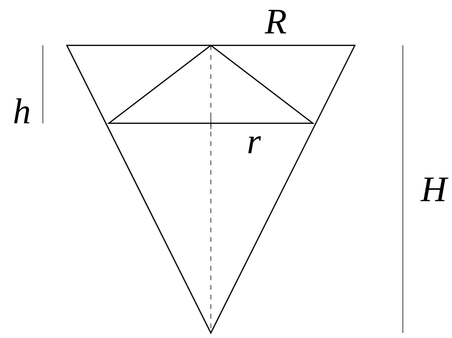
$$\begin{aligned} d^2 &= x^2 + y^2 \\ 2d\,\frac{dd}{dt} &= 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \\ 2d\,\frac{dd}{dt} &= 4x + 2y \end{aligned}$$

After 2 seconds, we know that $x = 4$, $y = 5$, and $z = \sqrt{41}$. So

$$\begin{aligned} 2\sqrt{41} \cdot \frac{dd}{dt} &= 26 \\ \frac{dd}{dt} &= \frac{13}{\sqrt{41}} \end{aligned}$$

After 2 seconds, the distance between the girl and the balloon is increasing at $\frac{13}{\sqrt{41}}$ m/sec. □

6. (20 points) Suppose you have a cone with *constant* height $H = 3$ and *constant* radius $R = 1$, and you want to put a smaller cone "upside down" inside the larger cone (see the picture). If h is the height of the smaller cone, what should h be to maximize the volume of the smaller cone? Recall that the volume of a cone with base radius r and height h is given by the formula $V = \frac{1}{3}\pi r^2 h$. (Hint: Use similar triangles to get the relationship between h and r.)



Solution.

We can do this without using the particular values for H and R.

Using similar triangles, we find that

$$\frac{r}{R} = \frac{H-h}{H},$$

or

$$r = \frac{R(H-h)}{H}.$$

Then, if V is the volume of the smaller cone,

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \left(\frac{R(H-h)}{H}\right)^2 h \\ &= \frac{\pi R^2}{3H^2} h(H-h)^2 \\ &= \frac{\pi R^2}{3H^2} (H^2 h - 2Hh^2 + h^3) \\ V'(h) &= \frac{\pi R^2}{3H^2} (H^2 - 4Hh + 3h^2) \\ &= \frac{\pi R^2}{3H^2} (H-3h)(H-h) \end{aligned}$$

So the critical points of $V(h)$ are $h = H$ and $h = \frac{H}{3}$. The domain of $V(h)$ is $[0, H]$. By the Closed Interval Method, the maximum value of $V(h)$ occurs at $h = 0$, $h = \frac{H}{3}$, or

$h = H$. Since $V(0) = V(H) = 0$, it follows that the volume is maximized at $h = H$. So $h = 3$. $\square$

7. (20 points) Compute the given limits, if they exist. If you assert that a limit does not exist, you need to justify your answer to get full credit.

(a) (5 points)

$$\lim_{x\to\infty}(\sqrt{x^2-3x+1}-\sqrt{x^2+2})$$

Solution.

$$\begin{aligned}
&\lim_{x\to\infty}(\sqrt{x^2-3x+1}-\sqrt{x^2+2})\\
=\ &\lim_{x\to\infty}\frac{\sqrt{x^2-3x+1}-\sqrt{x^2+2}}{1}\cdot\frac{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}\\
=\ &\lim_{x\to\infty}\frac{(x^2-3x+1)-(x^2+2)}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}\\
=\ &\lim_{x\to\infty}\frac{-3x-1}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}\\
=\ &\lim_{x\to\infty}\frac{-3x-1}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}\cdot\frac{1/x}{1/x}\\
=\ &\lim_{x\to\infty}\frac{-3-\frac{1}{x}}{\sqrt{1-\frac{3}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{2}{x^2}}}\\
=\ &-\frac{3}{2}
\end{aligned}$$

□

(b) (5 points)

$$\lim_{x\to 2}e^{\frac{1}{x-2}}$$

Solution.

$$\begin{aligned}
\lim_{x\to 2^+}e^{\frac{1}{x-2}} &= e^{\frac{1}{0^+}}=e^{\infty}=\infty\\
\lim_{x\to 2^-}e^{\frac{1}{x-2}} &= e^{\frac{1}{0^-}}=e^{-\infty}=0
\end{aligned}$$

Since the right- and left-hand limits differ, the limit does not exist. □

(c) (5 points)

$$\lim_{x\to\infty} \frac{x^3+1}{e^x}$$

Solution. Since

$$\lim_{x\to\infty} x^3+1=\infty$$

and

$$\lim_{x\to\infty} e^x=\infty$$

we are in a situation were we can apply L'Hopital's rule.

$$\lim_{x\to\infty} \frac{x^3+1}{e^x} = \lim_{x\to\infty} \frac{3x^2}{e^x}$$

We remain in a situation where L'Hopital's rule applies, and so we use it twice more:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3+1}{e^x} &= \lim_{x\to\infty} \frac{3x^2}{e^x} \\ &= \lim_{x\to\infty} \frac{6x}{e^x} \\ &= \lim_{x\to\infty} \frac{6}{e^x}\end{aligned}$$

Now we can evaluate the limit using normal methods: since the numerator tends to 6 while the denominator tends to ∞, the limit is 0.

$$\lim_{x\to\infty} \frac{x^3+1}{e^x}=0$$

□

(d) (5 points)

$$\lim_{x\to 2^+} \frac{2-x}{\ln(x-2)}$$

Solution. Since the numerator tends to 0 while the denominator tends to $-\infty$, the limit is 0. Note in particular that L'Hopital's rule does not apply.

□

8. (10 points) Estimate the following using linear approximation.

(a) (5 points)

$$\sin^2(\pi + .01)$$

Solution. Let $f(x) = \sin^2(x)$. Then we will linearly approximate f at $a = \pi$. The formula for the approximation is $L(x) = f(a) + f'(a) \cdot (x - a)$.

$f'(x) = 2\sin(x)\cos(x)$. Then $f(a) = 0$ and $f'(a) = 0$. Hence $L(x) = 0 + 0(x - \pi)$. Therefore, $L(\pi + .01) = 0$ and so we estimate $\sin^2(\pi + .01) \approx 0$.

□

(b) (5 points)

$$f(x) = x^3 - 2x + 1$$

Estimate f(-.05).

Solution. We will linearly approximate near 0. $f'(x) = 3x^2 - 2$. Therefore $f(0) = 1$ and $f'(0) = -2$. The linear approximation is $L(x) = 1 - 2(x - 0)$ and so $L(-.05) = 1 - 2(-.05) = 1.1$.

Therefore, $f(-.05) \approx 1.1$.

□

9. (5 points) – Do not attempt this question until completing the rest of the exam.

(a) State the intermediate value theorem.

Solution. Let f be a continuous function on the closed interval $[a, b]$. Then for every d between $f(a)$ and $f(b)$, there exists c in $[a, b]$ such that $f(c) = d$.

$\square$

(b) State the mean value theorem.

Solution. Let f be a continuous function on the closed interval $[a, b]$, differentiable on the open interval (a, b). Then there exists some c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

$\square$

(c) On July 4, 2008, Joey Chestnut ate 59 hot-dogs in 10 minutes to win Nathan's Hot Dog Eating Contest. Assume that the number of hot dogs that he consumes is a differentiable function of time and that at at the start of the competition, his rate of consumption is 0 hot dogs per minute. Give a precise argument that he was eating at a rate of 4 hot dogs per minute at some point during the competition.

Solution. Let f denote the number of hot-dogs that Mr. Chestnut has consumed with $f(0) = 0$ denoting the start of the competition. Then $f(10) = 59$. By the mean value theorem (applied to f and the interval $[0, 10]$), for some c in $(0, 10)$, $f'(c) = 5.9$. Since we are told that $f'(0) = 0$, the intermediate value theorem (applied to f', the interval $[0, c]$, and the number 4 which is between $f'(0) = 0$ and $f'(c) = 5.9$) says that there is some $\tilde{c}$ in $[0, c]$ such that $f'(\tilde{c}) = 4$.

Therefore, Mr. Chestnut was eating at a rate of 4 hot dogs per minute at some point during the competition.

$\square$