PRACTICE FINAL SOLUTIONS

1. True/False Questions. No explanation is needed.

(1) If $f'(x) < 0$ for $1 < x < 6$, then $f(x)$ is decreasing on $(1, 6)$.
(2) If $f(x)$ has an local minimum value at $x = c$, then $f'(c) = 0$.
(3) $f'(x)$ has the same domain as $f(x)$.
(4) If both $f(x)$ and $g(x)$ are differentiable, then $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}g(x)$.
(5) A function has at most two vertical asymptotes.

Solution.

(1) True. $f'(x) < 0$ means $f(x)$ is decreasing.
(2) False. It's possible that $f'(c)$ does not exist. Example: $f(x) = |x|$ and $c = 0$.
(3) False. The domain of $f'(x)$ can be smaller than the domain of $f(x)$. Example: $f(x) = |x|$.
(4) False. $\frac{\mathrm{d}}{\mathrm{d}x}(fg(x)) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}g(x) + \frac{\mathrm{d}g(x)}{\mathrm{d}x}f(x)$.
(5) False. The function $\tan x$ has infinitely many vertical asymptotes.

2. Find the following limits, or explain why one does not exist. If the limit involves infinity, explain whether it is ∞ or $-\infty$.

(1) $\lim_{x \to 0^+} \frac{\cos x}{x}$
(2) $\lim_{x \to \infty} \frac{\sqrt{x^2 - 6}}{x + 6}$
(3) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$
(4) $\lim_{x \to 0} (e^x + x)^{\frac{1}{x}}$

Solution.

(1) $\lim_{x \to 0^+} \frac{\cos x}{x}$ is of the type $\frac{1}{0^+}$ therefore it's ∞.

(2)

$$\lim_{x\to\infty}\frac{\sqrt{x^2-6}}{x+6}=\lim_{x\to\infty}\frac{x\frac{\sqrt{x^2-6}}{x}}{x(1+\frac{6}{x})}=\lim_{x\to\infty}\frac{\sqrt{1-\frac{6}{x^2}}}{1+\frac{6}{x}}=\frac{\sqrt{1}}{1}=1$$

(3)

$$\begin{aligned}\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)&=\lim_{x\to 0^+}\frac{\sin x-x}{x\sin x} &&\text{plug in } x=\infty\Rightarrow\frac{0}{0}\\ &\underset{L}{=}\lim_{x\to 0^+}\frac{\cos x-1}{\sin x+x\cos x} &&\text{plug in } x=\infty\Rightarrow\frac{0}{0}\\ &\underset{L}{=}\lim_{x\to 0^+}\frac{-\sin x}{\cos x-x\sin x+\cos x}\\ &=\frac{0}{1}=0\end{aligned}$$

(4) Because $(e^x+x)^{\frac{1}{x}}=e^{\frac{1}{x}\cdot\ln(e^x+x)}$, we first compute

$$\lim_{x\to 0}\frac{1}{x}\cdot\ln(e^x+x)=\lim_{x\to 0}\frac{\ln(e^x+x)}{x}\underset{L}{=}\lim_{x\to 0}\frac{\frac{1}{e^x+x}(e^x+1)}{1}=\frac{2}{1}=2\,.$$

Then the answer is e^2.

(5)

$$\ln h(x)=\ln\frac{(x^2-2)^3}{(x+3)^5\sqrt{x+1}}=3\ln(x^2-2)-5\ln(x+3)-\frac{1}{2}\ln(x+1)\,.$$

Differentiate both side with respect to x, we get

$$\frac{1}{h(x)}h'(x)=3\frac{2x}{x^2-2}-5\frac{1}{x+3}-\frac{1}{2}\frac{1}{x+2}$$

$$\begin{aligned}h'(x)=&h(x)\left(3\frac{2x}{x^2-2}-5\frac{1}{x+3}-\frac{1}{2}\frac{1}{x+2}\right)\\ =&\frac{(x^2-2)^3}{(x+3)^5\sqrt{x+1}}\left(\frac{6x}{x^2-2}-\frac{5}{x+3}-\frac{1}{2x+4}\right)\end{aligned}$$

3. Compute the following derivatives, using any method you like.

(1) $f(x)=\pi^{2x-7}+\sqrt{1-\sqrt{1-x^4}}$

(2) $g(x)=e^x\cdot(7x^2+\arcsin x^2)$

(3) $h(x)=\dfrac{(x^2-2)^3}{(x+3)^5\sqrt{x+1}}$

(4) $k(t)=\cos(t^{\frac{1}{t}})$

Solution.

(1)

$$\begin{aligned} f'(x) =& \pi^{2x-7} \cdot 2\ln\pi + \frac{1}{2\sqrt{(1-\sqrt{1-x^4})}}(1-\sqrt{1-x^4})' \\ =& \pi^{2x-7} \cdot 2\ln\pi + \frac{1}{2\sqrt{(1-\sqrt{1-x^4})}}\frac{4x^3}{2\sqrt{1-x^4}} \\ =& \pi^{2x-7} \cdot 2\ln\pi + \frac{x^3}{\sqrt{(1-\sqrt{1-x^4})(1-x^4)}} \end{aligned}$$

(2)

$$g'(x) = e^x(7x^2 + \arcsin x^2) + e^x(14x + \frac{2x}{\sqrt{1-x^4}})$$

(3)

$$\ln h(x) = \ln \frac{(x^2-2)^3}{(x+3)^5\sqrt{x+1}} = 3\ln(x^2-2) - 5\ln(x+3) - \frac{1}{2}\ln(x+1).$$

Differentiate both side with respect to x, we get

$$\frac{1}{h(x)}h'(x) = 3\frac{2x}{x^2-2} - 5\frac{1}{x+3} - \frac{1}{2}\frac{1}{x+2}$$

$$\begin{aligned} h'(x) =& h(x)\left(3\frac{2x}{x^2-2} - 5\frac{1}{x+3} - \frac{1}{2}\frac{1}{x+2}\right) \\ =& \frac{(x^2-2)^3}{(x+3)^5\sqrt{x+1}}\left(\frac{6x}{x^2-2} - \frac{5}{x+3} - \frac{1}{2x+4}\right) \end{aligned}$$

(4)

$$k'(t) = -\sin(t^{\frac{1}{t}}) \cdot (t^{\frac{1}{t}})'.$$

Let

$$y = t^{\frac{1}{t}},$$

then we have

$$\ln y = \frac{\ln t}{t}.$$

Differentiate both sides, we have

$$\frac{y'}{y} = \frac{1-\ln t}{t^2}.$$

Therefore

$$y' = t^{\frac{1}{t}} \cdot \frac{1-\ln t}{t^2}.$$

So

$$k'(t) = -\sin(t^{\frac{1}{t}}) \cdot t^{\frac{1}{t}} \cdot \frac{1-\ln t}{t^2}.$$

4. Answer the following questions.

(1) Complete the definition: a function $f(x)$ is differentiable at $x = a$ if ____________.
(2) Consider the function

$$f(x) = \begin{cases} x^3 \sin(\frac{1}{x}), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Use the above definition to decide whether $f(x)$ is differentiable at $x = 0$.

Solution.

(1) A function $f(x)$ is differentiable at $x = a$ if $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^3 \sin(\frac{1}{h}) - 0}{h} = \lim_{h \to 0} h^2 \sin(\frac{1}{h}) .$$

Because $-1 \leqslant \sin(\frac{1}{h}) \leqslant 1$, $-h^2 \leqslant h^2 \sin(\frac{1}{h}) \leqslant h^2$. The limits on both sides

$$\lim_{h \to 0} -h^2 = 0 = \lim_{h \to 0} h^2 .$$

The squeeze theorem imples $\lim_{h \to 0} h^2 \sin(\dfrac{1}{h}) = 0$, in particular the limit exists. Therefore $f(x)$ is differentiable at $x = 0$.

5. Answer the following questions.

(1) Give a precise statement of the Intermediate Value Theorem.
(2) Use the Intermediate Value Theorem to show that there exists a solution to the equation

$$\ln x = \sin\left(\frac{\pi}{2} x\right)$$

on the interval $(0, \infty)$.

Solution.

(1) Please find it on page 120 in text.
(2) Let

$$f(x) = \ln x - \sin\left(\frac{\pi}{2} x\right) .$$

Note that both $\ln x$ and $\sin\left(\frac{\pi}{2} x\right)$ are continuous functions, so $f(x)$ is also a continuous function. To use the Intermediate Value Theorem to prove the equation has a solution, we just need to find two numbers a and b in the interval $(0, \infty)$, such that $f(a)$ and $f(b)$ have different signs.

There are many different choices for a and b. The following is a possible way. Note that

$$f(1) = \ln 1 - \sin\left(\frac{\pi}{2}\right) = 0 - 1 = -1 < 0,$$

and

$$f(e^2) = \ln e^2 - \sin\left(\frac{\pi}{2}e^2\right) = 2 - \sin\left(\frac{\pi}{2}e^2\right) \geqslant 2 - 1 = 1 > 0.$$

Therefore, if we apply the Intermediate Value Theorem to the function $f(x)$ on the closed interval $[1, e^2]$, we can conclude that there exists a number c between 1 and e^2, such that $f(c) = 0$, that is,

$$\ln c = \sin\left(\frac{\pi}{2}c\right)$$

as desired.

6. Answer the following questions.

(1) Give a precise statement of the Mean Value Theorem.
(2) Let f be a differentiable function such that $f(0) = 0$ and $f'(x) \leqslant 1$ for all x. Use the Mean Value Theorem to show that $f(2) \neq 3$.

Solution.

(1) Please find on page 272 in text.
(2) Suppose $f(2) = 3$, by the Mean Value Theorem we know there is a c in $(0, 2)$ satisfies

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{3}{2}.$$

This contradicts the fact that $f'(x) \leqslant 1$ for all x. Thus $f(2) \neq 3$.

7. The equation $x^2y^2 + xy = 2$ describe a curve in the xy-plane.

(1) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.
(2) Find the equation of the line tangent to the curve at the point $(-1, 2)$.
(3) Find the coordinates (x, y) of all points on the curve where the tangent line is parallel to the line $x + y = 1$.

Solution.

(1) Use implicit differentiation, we differentiate both sides of the equation with respect to x.

$$x^2y^2 + xy = 2 \Rightarrow 2xy^2 + x^2(2y)y' + y + xy' = 0$$
$$\Rightarrow y'(2x^2y + x) = -2xy^2 - y \Rightarrow y' = \frac{-2xy^2 - y}{2x^2y + x}$$

(2) At $(-1, 2)$ $y' = \frac{6}{4-1} = 2$. The equation of the tangent line is

$$y - 2 = 2(x + 1) \Rightarrow y = 2x + 4$$

(3) Because the slope of $x+y = 1$ is -1, if a tangent line is parallel to $x+y = 1$ the tangent line also has slope -1.

$$-1 = \frac{-2xy^2 - y}{2x^2y + x} \Rightarrow 2xy^2 + y = 2x^2y + x$$
$$\Rightarrow y(2xy + 1) = x(2xy + 1) \Rightarrow (x - y)(2xy + 1) = 0$$

Thus either $xy = -\frac{1}{2}$ or $x = y$. If $xy = -\frac{1}{2}$, $x^2y^2 + xy = \left(-\frac{1}{2}\right) - \frac{1}{2} = -\frac{1}{4} \neq 2$ which is impossible, which means $x = y$. Plug in $x = y$ into $x^2y^2 + xy = 2$,

$$x^4 + x^2 = 2 \Rightarrow x^4 + x^2 - 2 = 0 \Rightarrow (x^2 + 2)(x^2 - 1) = 0 \Rightarrow x^2 = 1$$

because $x^2 + 2 > 0$. Therefore the answer is $(1, 1)$ and $(-1, -1)$.

8. Consider the function $f(x) = x^{\frac{2}{3}}$.

(1) Find the linear approximation of the function $f(x)$ at the point $a = 8$; that is, find the linear function $L(x)$ that best approximate $f(x)$ for values of x near 8.

(2) Use the above linear approximation to estimate $(8.04)^{\frac{2}{3}}$. Is your approximation an overestimate or an underestimate of the actual value? Explain fully.

Solution.

(1) Since

$$f(x) = x^{\frac{2}{3}},$$

we have

$$f(8) = 8^{\frac{2}{3}} = 4.$$

Since

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}},$$

we have

$$f'(8) = \frac{2}{3}8^{-\frac{1}{3}} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

Thus,

$$L(x) = f(8) + f'(8) \cdot (x - 8) = 4 + \frac{1}{3}(x - 8).$$

(2) $(8.04)^{\frac{2}{3}}$ is $f(8.04)$. Since 8.04 is near 8, we can say

$$f(8.04) \approx L(8.04) = 4 + \frac{1}{3}(8.04 - 8) = 4 + \frac{1}{3} \cdot 0.04 \approx 4.013.$$

Since

$$f''(x) = \frac{2}{3} \cdot \left(-\frac{1}{3}\right) x^{-\frac{4}{3}} < 0,$$

we know that the graph of $f(x)$ is concave down when x is near 8. Therefore the tangent line lies above the graph, so that the values of $L(x)$ will be overestimates of the values of $f(x)$.

9. Consider the function $f(x) = x^{\frac{1}{3}}(x - 8)^2$.

(1) Find all critical numbers of f.
(2) Find the absolute maximum and minimum values of f on the interval $[-1, 8]$.

Solution.

(1) We first find the derivative function.

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-\frac{2}{3}}(x-8)^2 + x^{\frac{1}{3}} \cdot 2(x-8) \\ &= \frac{(x-8)^2 + 6x(x-8)}{3x^{\frac{2}{3}}} \\ &= \frac{(x-8)(7x-8)}{3x^{\frac{2}{3}}} \end{aligned}$$

The critical numbers are the numbers at which $f'(x)$ equals 0 or undefined. Therefore all critical numbers are $x = 8, x = \frac{8}{7}, x = 0$.

(2) To find the absolute maximum and minimum values of f on the interval $[-1, 8]$, we use the closed interval method. We need to compute the values of f at all critical numbers and endpoints, which are -1, 0, $\frac{8}{7}$, 8.

$$f(-1) = (-1)^{\frac{1}{3}}(-1-8)^2 = (-1) \cdot 81 = -81;$$

$$f(0) = 0;$$

$$f\left(\frac{8}{7}\right) = \left(\frac{8}{7}\right)^{\frac{1}{3}} \left(\frac{8}{7} - 8\right)^2 > 0;$$

$$f(8) = 0.$$

Therefore the absolute minimum value is $f(-1) = -81$, and the absolute maximum value is $f\left(\frac{8}{7}\right) = \left(\frac{8}{7}\right)^{\frac{1}{3}} \left(\frac{8}{7} - 8\right)^2$. (You don't have to simplify your answer.)

10. Consider the function $f(x) = \dfrac{x^2}{x^2 - 1}$.

(1) Find the domain and zeroes of $f(x)$.
(2) Find all horizontal and vertical asymptotes of $f(x)$. Justify your answer by limit computations.
(3) Find $f'(x)$ and $f''(x)$, using any method you like.
(4) Find the intervals of increase and decrease.
(5) Find all local maximum and local minimum values.
(6) Find the intervals of concavity and all inflection points.
(7) Use the information from all above parts to sketch the graph of $f(x)$.

Solution.

(1) Domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ (you can also say all real numbers except -1 and 1). Zeroes are the numbers at which $f(x) = 0$. The only such number is $x = 0$.
(2) To find the horizontal asymptotes, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^2}} = \frac{1}{1-0} = 1$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{x^2-1} = \lim_{x\to-\infty} \frac{1}{1-\frac{1}{x^2}} = \frac{1}{1-0} = 1.$$

So the only horizontal asymptote is $y = 1$.

The possible candidates of vertical asymptotes are zeroes of the denominator, i.e., $x = -1$ and $x = 1$. We use the one-sided limits to verify they are indeed vertical asymptotes. When x approaches -1 from the left side, x^2 approaches 1 from the right side and $x^2 - 1$ approaches 0 from the right side, so

$$\lim_{x\to-1^-} \frac{x^2}{x^2-1} = \infty.$$

Similarly, we have

$$\lim_{x\to-1^+} \frac{x^2}{x^2-1} = -\infty;$$

$$\lim_{x\to1^-} \frac{x^2}{x^2-1} = -\infty;$$

$$\lim_{x\to1^+} \frac{x^2}{x^2-1} = \infty.$$

The limits all involve infinity, so $x = -1$ and $x = 1$ are vertical asymptotes.

(3) We have

$$f'(x) = \frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2};$$

$$\begin{aligned} f''(x) &= \frac{(-2)(x^2-1)^2 - (-2x)\cdot 2(x^2-1)(2x)}{(x^2-1)^4} \\ &= \frac{-2(x^2-1)^2 + 8x^2(x^2-1)}{(x^2-1)^4} \\ &= \frac{-2(x^2-1) + 8x^2}{(x^2-1)^3} \\ &= \frac{6x^2+2}{(x^2-1)^3}. \end{aligned}$$

(4) The numbers at which the derivative function $f'(x)$ is equal to 0 or undefined are -1, 0 and 1. So we need to consider the sign of $f'(x)$ on the four intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$. On the first two intervals $(-\infty, -1)$ and $(-1, 0)$, the numerator $-2x > 0$ and the denominator $(x^2-1)^2 > 0$, so $f'(x) > 0$. On the last two intervals $(0, 1)$ and $(1, \infty)$, the numerator $-2x < 0$ and the denominator $(x^2-1)^2 > 0$, so $f'(x) < 0$.

In summary, f is increasing on intervals $(-\infty, -1)$ and $(-1, 0)$, and decreasing on intervals $(0, 1)$ and $(1, \infty)$.

(5) First we need to find all critical numbers which are in the domain of the function. In part (d) we see that the numbers at which $f'(x)$ is equal to 0 or undefined are -1, 0 and 1. However -1 and 1 are not in the domain of f. So the only critical number is 0. We also know from part (d) that $f(x)$ is increasing when x is to the left of 0 and decreasing when x is to the right of 0. By First Derivative Test, the function has a local maximum at $x = 0$. The corresponding local maximum value is

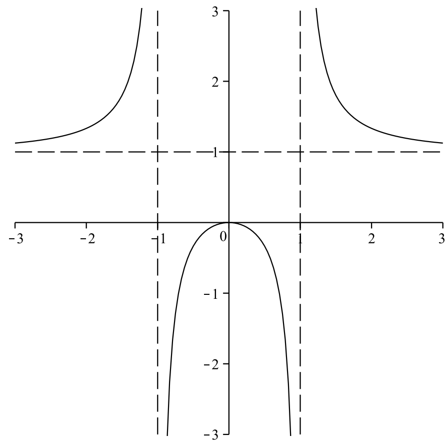
$$y = f(0) = \frac{0}{0-1} = 0.$$

(6) To find the intervals of concavity we need to analyze the sign of $f''(x)$. Note that the numerator in $f''(x)$ is $6x^2 + 2$ which is always positive. So the points at which $f''(x)$ could possibly change its sign are the zeroes of the denominator, which are -1 and 1. So we need to look at the sign of $f''(x)$ on three intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$. On intervals $(-\infty, -1)$ and $(1, \infty)$, the denominator $(x^2-1)^3 > 0$, so $f''(x) > 0$. On the interval $(-1, 1)$, the denominator $(x^2-1)^3 < 0$, so $f''(x) < 0$.

In summary, f is concave up on intervals $(-\infty, -1)$ and $(1, \infty)$, and concave down on the interval $(-1, 1)$.

The function f doesn't have any inflection points because neither -1 nor 1 is in the domain of f.

(7) The graph is as follows.



11. Harry Potter, a 5-foot-tall man, notices a small UFO on the ground, located 40 feet from where he stands in a flat field. The UFO suddenly begins a rapid vertical ascent, at a rate of 10 feet per second. Throughout the ascent, a bright light on the ship illuminates the entire field below, casting a shadow of Harry onto the ground. What is the rate of change of the length of Harry's shadwo exactly three seconds after the UFO has taken off? (Hint: at any moment, the head of Harry's shadow is always located on the ground, and on the line determined by the UFO's light and Harry's head.)

Solution. Please refer to the last page for a picture illustrating the situation.

In the picture, we use x for the length of the shadow at a given moment, and y for the height of the UFO.

By similar triangles, we have

$$\frac{x}{40+x} = \frac{5}{y},$$

so that

$$xy = 5(40+x).$$

Differentiating with respect to time, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(xy) = \frac{\mathrm{d}}{\mathrm{d}t}(5(40+x)),$$

$$y\frac{\mathrm{d}x}{\mathrm{d}t} + x\frac{\mathrm{d}y}{\mathrm{d}t} = 5\frac{\mathrm{d}x}{\mathrm{d}t}.$$

When 3 seconds has passed, we have $y = 30$, so that

$$30x = 5(40+x),$$

so $x = 8$. Also we know $\frac{\mathrm{d}y}{\mathrm{d}t} = 10$. Thus, at our moment of interest, we have

$$30\frac{\mathrm{d}x}{\mathrm{d}t} + 8 \cdot 10 = 5\frac{\mathrm{d}x}{\mathrm{d}t},$$

so that

$$25 \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = -80,$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{80}{25} \text{ feet/second.}$$

That is, Harry's shadow is decreasing in length at a rate of $\frac{80}{25}$ feet/second.

12. In the xy-plane, any negatively-sloped line that passes through the point $(2, 3)$ will form a right triangle with the x-axis and y-axis in the first quadrant. Among all possible such lines (negative slope, passing through (2,3)), find the equation of the line that forms a triangle of minimal area. Justify completely.

Solution. Please refer to the last page for a picture illustrating the situation.

The line with slope m through $(2, 3)$ has an equation

$$y - 3 = m(x - 2).$$

The triangle it cuts off has area $\frac{1}{2}ab$, where a is the x-intercept and b is the y-intercept. That is, a satisfies

$$0 - 3 = m(a - 2),$$

$$a = 2 - \frac{3}{m};$$

and b satisfies

$$b - 3 = m(0 - 2),$$

$$b = 3 - 2m.$$

Thus, in terms of slope m, the quantity $Q(m)$ we wish to minimize is

$$\begin{aligned} Q(m) &= \text{area} = \frac{1}{2}ab \\ &= \frac{1}{2}\left(2 - \frac{3}{m}\right)(3 - 2m) \\ &= \frac{1}{2}\left(6 - \frac{9}{m} - 4m + 6\right) \\ &= 6 - 2m - \frac{9}{2m} \end{aligned}$$

where the domain for $Q(m)$ is $m < 0$, that is, the open interval $(-\infty, 0)$. To find the critical numbers of Q, we find

$$Q'(m) = -2 + \frac{9}{2m^2}.$$

Note that Q' is never undefined on the domain $(-\infty, 0)$. So we are left with $Q' = 0$,

$$-2 + \frac{9}{2m^2} = 0,$$
$$m^2 = \frac{9}{4},$$
$$m = \pm\frac{3}{2},$$

though only $m = -\frac{3}{2}$ is in the domain $(-\infty, 0)$.

We can use the Second Derivative Test for Absolute Extrema. Since

$$Q''(m) = \frac{9}{2} \cdot (-2m^{-3}) = -\frac{9}{m^3},$$

we have that $Q''(m) > 0$ for all $m < 0$, and thus the single critical number $m = -\frac{3}{2}$ is a point of absolute minimum for Q on this domain. The corresponding line is

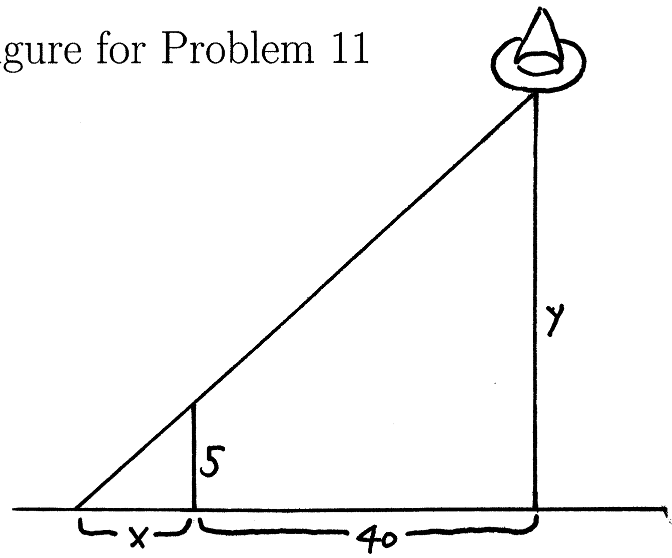
$$y - 3 = -\frac{3}{2}(x - 2).$$

(Triangle area is

$$Q\left(-\frac{3}{2}\right) = 12,$$

though that was not asked for.)

• Figure for Problem 11



• Figure for Problem 12