PRACTICE FINAL

1. True/False Questions. No explanation is needed.

(1) If $f'(x) < 0$ for $1 < x < 6$, then $f(x)$ is decreasing on $(1, 6)$.
(2) If $f(x)$ has an local minimum value at $x = c$, then $f'(c) = 0$.
(3) $f'(x)$ has the same domain as $f(x)$.
(4) If both $f(x)$ and $g(x)$ are differentiable, then $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}g(x)$.
(5) A function has at most two vertical asymptotes.

2. Find the following limits, or explain why one does not exist. If the limit involves infinity, explain whether it is ∞ or $-\infty$.

(1) $\lim_{x \to 0^+} \frac{\cos x}{x}$
(2) $\lim_{x \to \infty} \frac{\sqrt{x^2 - 6}}{x + 6}$
(3) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$
(4) $\lim_{x \to 0} (e^x + x)^{\frac{1}{x}}$

3. Compute the following derivatives, using any method you like.

(1) $f(x) = \pi^{2x-7} + \sqrt{1 - \sqrt{1 - x^4}}$
(2) $g(x) = e^x \cdot (7x^2 + \arcsin x^2)$
(3) $h(x) = \frac{(x^2 - 2)^3}{(x + 3)^5 \sqrt{x + 1}}$
(4) $k(t) = \cos(t^{\frac{1}{t}})$

4. Answer the following questions.

(1) Complete the definition: a function $f(x)$ is differentiable at $x = a$ if ____________.

(2) Consider the function

$$f(x) = \begin{cases} x^3 \sin(\frac{1}{x}), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Use the above definition to decide whether $f(x)$ is differentiable at $x = 0$.

5. Answer the following questions.

(1) Give a precise statement of the Intermediate Value Theorem.
(2) Use the Intermediate Value Theorem to show that there exists a solution to the equation

$$\ln x = \sin\left(\frac{\pi}{2}x\right)$$

on the interval $(0, \infty)$.

6. Answer the following questions.

(1) Give a precise statement of the Mean Value Theorem.
(2) Let f be a differentiable function such that $f(0) = 0$ and $f'(x) \leqslant 1$ for all x. Use the Mean Value Theorem to show that $f(2) \neq 3$.

7. The equation $x^2y^2 + xy = 2$ describe a curve in the xy-plane.

(1) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.
(2) Find the equation of the line tangent to the curve at the point $(-1, 2)$.
(3) Find the coordinates (x, y) of all points on the curve where the tangent line is parallel to the line $x + y = 1$.

8. Consider the function $f(x) = x^{\frac{2}{3}}$.

(1) Find the linear approximation of the function $f(x)$ at the point $a = 8$; that is, find the linear function $L(x)$ that best approximate $f(x)$ for values of x near 8.
(2) Use the above linear approximation to estimate $(8.04)^{\frac{2}{3}}$. Is your approximation an overestimate or an underestimate of the actual value? Explain fully.

9. Consider the function $f(x) = x^{\frac{1}{3}}(x-8)^2$.

(1) Find all critical numbers of f.
(2) Find the absolute maximum and minimum values of f on the interval $[-1, 8]$.

10. Consider the function $f(x) = \dfrac{x^2}{x^2-1}$.

(1) Find the domain and zeroes of $f(x)$.
(2) Find all horizontal and vertical asymptotes of $f(x)$. Justify your answer by limit computations.
(3) Find $f'(x)$ and $f''(x)$, using any method you like.
(4) Find the intervals of increase and decrease.
(5) Find all local maximum and local minimum values.
(6) Find the intervals of concavity and all inflection points.
(7) Use the information from all above parts to sketch the graph of $f(x)$.

11. Harry Potter, a 5-foot-tall man, notices a small UFO on the ground, located 40 feet from where he stands in a flat field. The UFO suddenly begins a rapid vertical ascent, at a rate of 10 feet per second. Throughout the ascent, a bright light on the ship illuminates the entire field below, casting a shadow of Harry onto the ground. What is the rate of change of the length of Harry's shadwo exactly three seconds after the UFO has taken off? (Hint: at any moment, the head of Harry's shadow is always located on the ground, and on the line determined by the UFO's light and Harry's head.)

12. In the xy-plane, any negatively-sloped line that passes through the point $(2, 3)$ will form a right triangle with the x-axis and y-axis in the first quadrant. Among all possible such lines (negative slope, passing through (2,3)), find the equation of the line that forms a triangle of minimal area. Justify completely.