Math 19 Calculus Summer 2008

Midterm
Wednesday, July 23

Name: ______________________________

I agree to abide by the honor code:

Signature: ______________________________

- You have 2 hours (7:00 – 9:00).
- No notes, books, or calculators are permitted.
- **You must show all work to receive credit!**
- Please only use techniques and theorems studied in this class.
- Please check your solutions carefully.

1.	__________	(/10 points)
2.	__________	(/10 points)
3.	__________	(/10 points)
4.	__________	(/10 points)
5.	__________	(/10 points)
6.	__________	(/10 points)
7.	__________	(/10 points)
8.	__________	(/10 points)
9.	__________	(/10 points)
10.	__________	(/10 points)
Total.	__________	(/100 points)

1. Answer the following questions with true or false. No explanation is needed.

T (a) The product rule states that $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

F (b) The quotient rule states that $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)+f(x)g'(x)}{[g(x)]^2}$.

F (c) $x < 0$ or $x > 1$ is written in interval notation as $(-\infty, 0] \cup [1, \infty)$.

T (d) If the derivative of a function $f(x)$ is negative on some interval (a, b), then $f(x)$ is decreasing on the interval (a, b).

F (e) All continuous functions are differentiable at all points in their domains.

F (f) The $\lim_{x\to\infty} \sin x$ exists and equals some real number L.

T (g) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist as finite limits, then $\lim_{x\to a}(f(x)g(x))$ must exist, and $\lim_{x\to a}(f(x)g(x)) = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$.

T (h) The second derivative of $f(x) = cx$ where c is a constant, always equals 0.

F (i) The $\lim_{x\to 2} \frac{x^2-4}{x-2}$ does not exist because $\frac{0}{0}$ is an indeterminate form.

T (j) The nth derivative of $f(x) = e^x$ is e^x.

2. (a) Compute

$$\lim_{x\to\infty} \arctan(\ln x)$$

$$\lim_{x\to\infty} \ln x = \infty$$

$$\lim_{y\to\infty} \arctan(y) = \frac{\pi}{2}$$

$$\lim_{x\to\infty} \arctan(\ln x) = \frac{\pi}{2}$$

(b) Compute

$$\lim_{x\to 4} \frac{x-\sqrt{3x+4}}{4-x}.$$

$$= \lim_{x\to 4} \left(\frac{x-\sqrt{3x+4}}{4-x}\right)\left(\frac{x+\sqrt{3x+4}}{x+\sqrt{3x+4}}\right)$$

$$= \lim_{x\to 4} \frac{x^2-3x-4}{(4-x)(x+\sqrt{3x+4})}$$

$$= \lim_{x\to 4} \frac{(x-4)(x+1)}{-(x-4)(x+\sqrt{3x+4})}$$

$$= \lim_{x\to 4} -\frac{(x+1)}{x+\sqrt{3x+4}}$$

$$= -\frac{4+1}{4+\sqrt{3\cdot 4+4}}$$

$$= -\frac{5}{8}$$

(c) Compute

$$\lim_{x\to\infty} \frac{5x^3+9x-4}{\sqrt{4x^6-x^3+8}}.$$

$$= \lim_{x\to\infty} \left(\frac{5x^3+9x-4}{\sqrt{4x^6-x^3+8}}\right)\left(\frac{1/x^3}{1/x^3}\right)$$

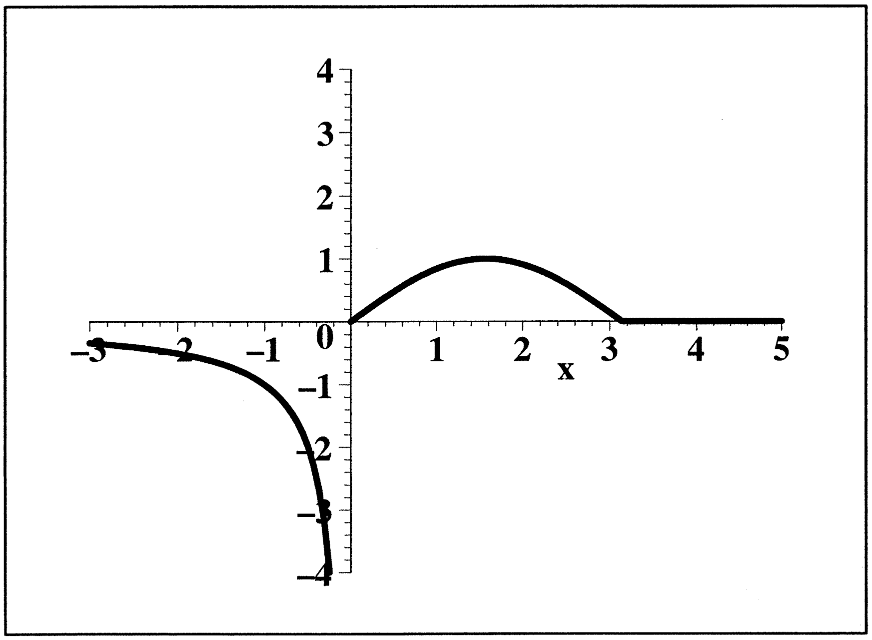
$$= \frac{\lim_{x\to\infty} 5+9x^{-2}-4x^{-3}}{\lim_{x\to\infty} \sqrt{4-x^{-3}+8x^{-6}}}$$

$$= \lim_{x\to\infty} \frac{5+9x^{-2}-4x^{-3}}{\sqrt{4-x^{-3}+8x^{-6}}}$$

$$= \frac{5}{\sqrt{4}}$$

$$= \frac{5}{2}$$

3. Given the following function $f(x)$



(a) Compute the following limits

$\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to 0^+} f(x) = 0$

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0} f(x)$ does not exist

(b) State the numbers at which $f(x)$ is not differentiable.

$x = 0, \ 3.1$

4. Let $f(x) = \frac{1}{x}$.

(a) Using the limit definition of the derivative, find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \frac{-1}{x^2}$$

(b) Check your answer by calculating the derivative in another way: state and apply a differentiation rule.

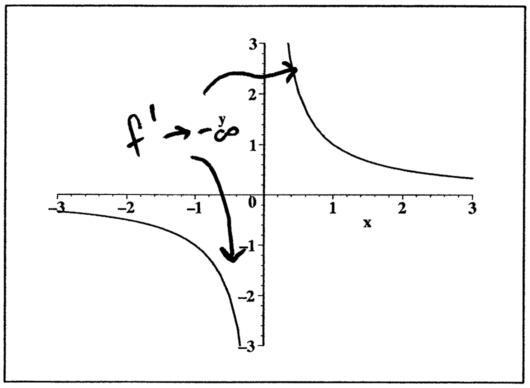
The power rule: $\frac{d}{dx}(x^p) = px^{p-1}$

Therefore, $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2}$

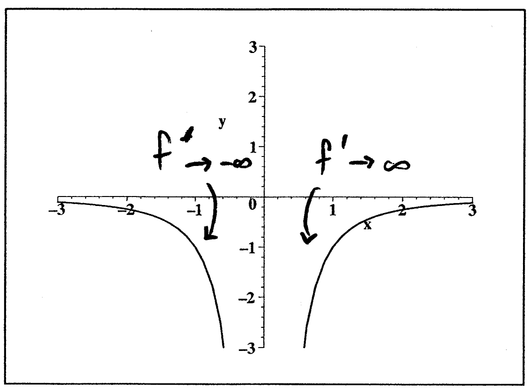
$-x^{-2} = \frac{-1}{x^2}$ so the answer in (a)

is correct.

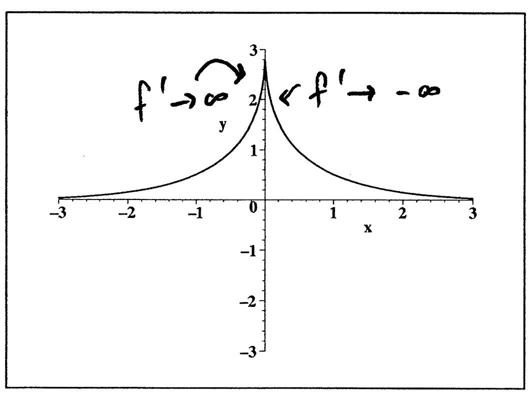
5. The graphs of three functions are drawn in the left column. The graphs of their derivatives are drawn in the right column. Match each function with its derivative. You do not need to justify your answer.



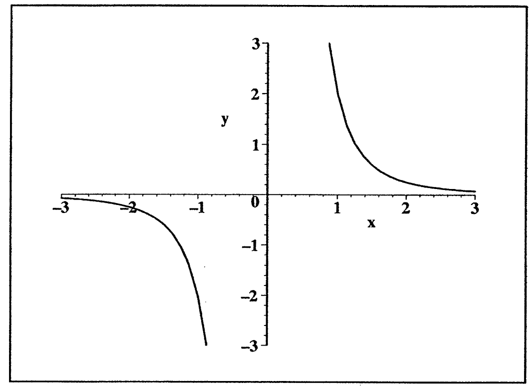
(A) $\longrightarrow$ (c)



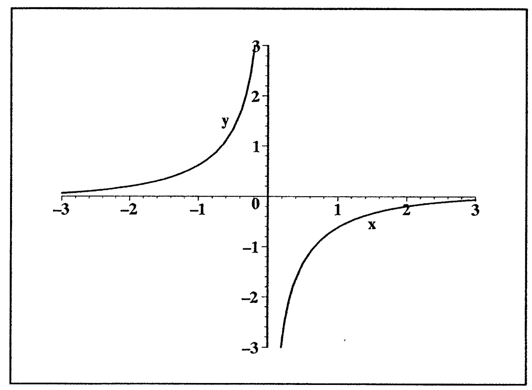
(B) $\longrightarrow$ (a)



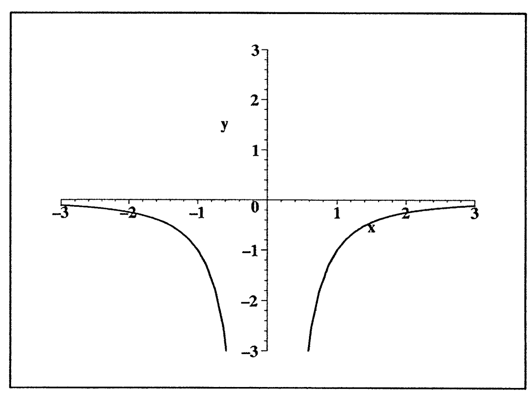
(C) $\longrightarrow$ (b)



(a)



(b)



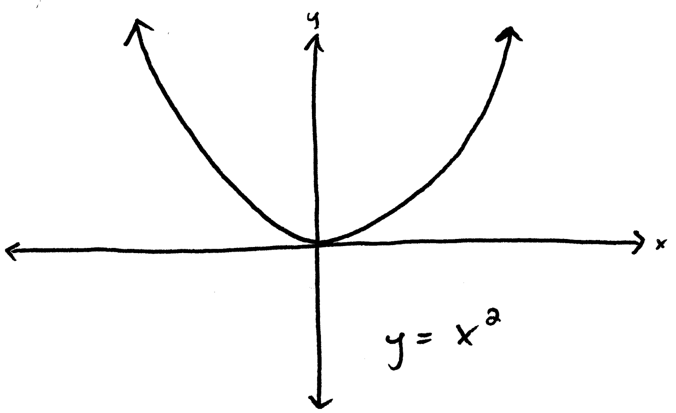
(c)

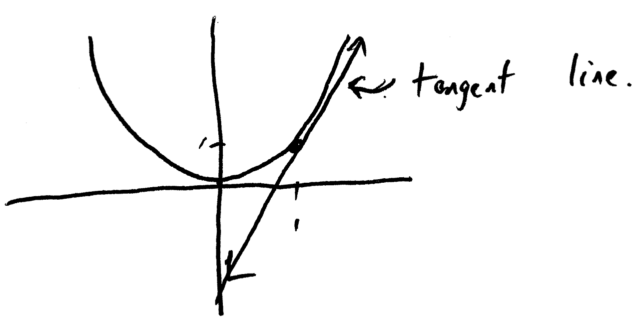
6. The derivative of a function $f(x)$ is defined as a limit of certain quotients.

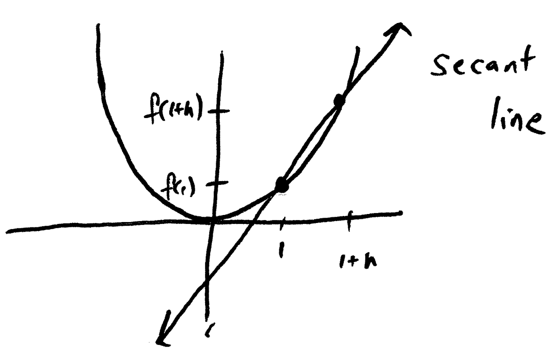
(a) Write down a limit definition of $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Let $f(x) = x^2$. Graph $f(x)$ and explain what $f'(1)$ represents and what the numbers $\frac{f(1+h)-f(1)}{h}$ represent geometrically.







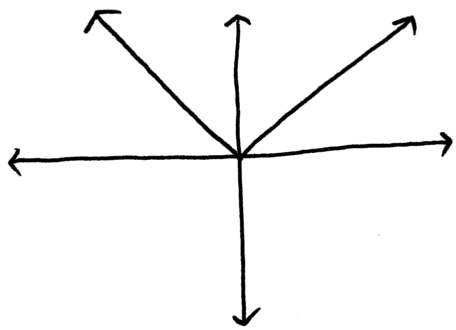
$f'(1)$ represents the slope of the tangent line at 1.

The difference quotient $\frac{f(1+h) - f(1)}{h}$ gives the slope of a secant line passing through $(1, f(1))$ and $(1+h, f(1+h))$.

For h small, it is close to $f'(1)$, since f is a differentiable function.

(c) Graph an example of a continuous function that is *not* differentiable at $x = 0$ and give a geometric argument why the derivative fails to exist at that point.

$f(x) = |x|$ is continuous but not differentiable at $x = 0$ because it has a "corner", as shown in the graph.



(Other acceptable answers would include continuous functions with cusps or vertical tangent lines, or talking directly about slopes of tangent lines from the left and right of zero not agreeing.)

7. Calculate $f'(x)$ when:

(a) $f(x) = \sin(x)\tan(x)$

$$f'(x) = (\sin(x))' \tan(x) + \sin(x) (\tan(x))'$$

$$= \cos(x) \tan(x) + \sin(x) (\sec(x))^2$$

$$= \sin(x) (1 + \sec^2(x))$$

There are other methods, e.g. $f(x) = \dfrac{\sin(x)^2}{\cos(x)}$

and use the quotient rule.

(b) $f(x) = e^{2x}\cos(2x) + 10x^5\sin(x)$

$$f'(x) = 2e^{2x} \cos(2x) + e^{2x} (-2\sin(2x))$$
$$+ 10 (5x^4 \sin(x) + x^5 \cos(x))$$

$$= 2e^{2x} (\cos(2x) - \sin(2x))$$
$$+ 10x^4 (5 \sin(x) + x \cos(x))$$

8. Let $f(x) = \frac{e^x}{x+1}$

(a) Calculate $f'(x)$.

quotient rule: $\left(\frac{g}{h}\right)' = \frac{hg' - gh'}{h^2}$

$$f'(x) = \frac{(x+1)e^x - e^x(1)}{(x+1)^2}$$

$$= \frac{xe^x}{(x+1)^2}$$

Note: it is best to simplify f' before calculating f''.

(b) Calculate $f''(x)$.

$$f''(x) = \frac{(x+1)^2\left(e^x + xe^x\right) - xe^x\left(2(x+1)\right)}{(x+1)^4}$$

$$= \frac{(x+1)^3 e^x - 2x(x+1)e^x}{(x+1)^4}$$

$$= \frac{\left((x+1)^2 - 2x\right)e^x}{(x+1)^3}$$

$$= \frac{(x^2+1)e^x}{(x+1)^3}$$

9. Let $y = \sqrt{5 - 2x^2}$.

(a) Calculate $\frac{dy}{dx}$.

$$y = (5 - 2x^2)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(5 - 2x^2)^{-1/2}(-4x)$$

$$= -2x(5 - 2x^2)^{-1/2}$$

(b) Calculate $\frac{d}{dx}\cos(y(x))$.

chain rule: $\frac{d}{dx}\cos(y(x)) = -\sin(y(x))\frac{dy}{dx}$

$$= -\sin\left(\sqrt{5 - 2x^2}\right)\left(-2x(5 - 2x^2)^{-1/2}\right)$$

$$= \frac{2x\sin\left(\sqrt{5 - 2x^2}\right)}{\sqrt{5 - 2x^2}}$$

10. (a) Complete the definition: $f(x)$ is continuous at $x = a$ provided the following conditions hold:

$f(a)$ exists, $\lim_{x \to a} f(x)$ exists

and $\lim_{x \to a} f(x) = f(a)$.

(It was enough to say: $\lim_{x \to a} f(x) = f(a)$)

(b) Let $f(x) = \begin{cases} x^{\frac{1}{3}} \cos \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Is $f(x)$ continuous at 0? Give a rigorous justification for your answer.

Yes, $f(x)$ is continuous at 0. We will use the "squeeze theorem".

$\left| x^{1/3} \cos\left(\frac{1}{x^2}\right) \right| \leq \left| x^{1/3} \right| = |x|^{1/3}$ since $-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$, $x \neq 0$

There fore, $-|x|^{1/3} \leq x^{1/3} \cos\left(\frac{1}{x^2}\right) \leq |x|^{1/3}$ for all $x \neq 0$.

$$\lim_{x \to 0} -|x|^{1/3} = -\left(\lim_{x \to 0} |x| \right)^{1/3} = 0$$

$$\lim_{x \to 0} |x|^{1/3} = \left(\lim_{x \to 0} |x| \right)^{1/3} = 0$$

Since these are equal and equal to 0, we can apply the theorem to conclude $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^{1/3} \cos\left(\frac{1}{x^2}\right) = 0$.

Since $f(0) = 0$ by definition of f, $\lim_{x \to 0} f(x) = f(0)$.

There fore, f is continuous at 0.