PRACTICE MIDTERM SOLUTIONS

1. True/False Questions. No explanation is needed.

(1) $f(x) = |x-2|$ is one-to-one.
(2) $\lim_{x\to 5}\left(\frac{2x}{x-5} - \frac{10}{x-5}\right) = \lim_{x\to 5}\frac{2x}{x-5} - \lim_{x\to 5}\frac{10}{x-5}$
(3) A function can have infinitely many horizontal asymptotes.
(4) If f is continuous on $[0,2]$, then f is differentiable on $[0,2]$.
(5) The n-th derivative of $f(x) = e^{2x}$ is $2^n e^{2x}$.

Solution.

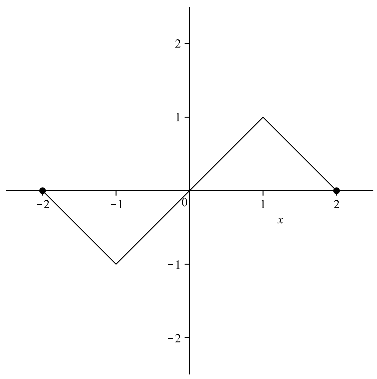
(1) False, because $f(3) = f(1) = 1$.
(2) False,
$$\lim_{x\to 5}\left(\frac{2x}{x-5} - \frac{10}{x-5}\right) = \lim_{x\to 5}\frac{2x-10}{x-5} = \lim_{x\to 5} 2 = 2.$$
However, neither $\lim_{x\to 5}\frac{2x}{x-5}$ nor $\lim_{x\to 5}\frac{10}{x-5}$ exists. So their difference makes no sense. Therefore the statement is false.
Note: this problem indicates that the subtraction rule is valid only when all limits exist.

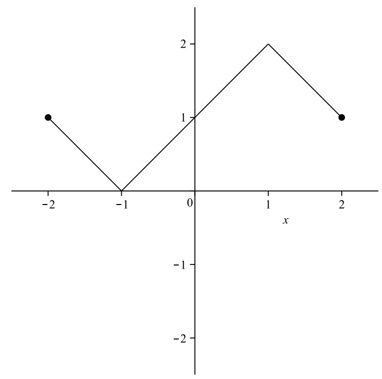
(3) False. A function $f(x)$ has horizontal asymptotes $x = a$ only if
$$\lim_{x\to\infty} f(x) = a \quad \text{or} \quad \lim_{x\to -\infty} f(x) = a\,.$$
Therefore a function can only have at most 2 horizontal asymptotes.
(4) False. The function $|x-1|$ is continuous at every point but it's not differentiable at 1.
(5) True. $f^{(1)}(x) = 2e^{2x} = 2^1e^{2x}$, $f^{(2)}(x) = 2\cdot 2e^{2x} = 2^2e^{2x}$. Because every time e^{2x} is differentiated, the result is multiplying it by 2, thus after differentiating e^{2x} n times, $f^{(n)}(x)$ becomes 2^ne^{2x}

2. The graph of $f(x)$ is shown. Answer the following questions and explain your reasoning:

(1) What is the domain of f?
(2) What is the range of f?
(3) Is f one-to-one?
(4) Where is f not differentiable?

(5) Sketch the graph of $-f(-x)+1$ on the coordinate system.





Solution.

(1) The domain of f is $[-2, 2]$.
(2) The range of f is $[-1, 1]$.
(3) f is not one-to-one. For example, there are three values of x, -2, 0 and 2, at which the function is 0.
(4) f is not differentiable at $x = -1$ and $x = 1$ because the graph has corners at these points. (Strictly speaking, at the end points $x = -2$ and $x = 2$, the function is not differentiable either, since the difference quotients at these points have only a one-sided limit.)
(5) See the graph on the right above.

3. Evaluate the following limits or show they do not exist.

(1) $\displaystyle\lim_{x\to -1} \frac{x^2 - 3x - 4}{x+1}$
(2) $\displaystyle\lim_{x\to \frac{1}{2}} \ln(\sin(\pi x))$
(3) $\displaystyle\lim_{x\to 2} (x^2-4)^2 \sin\left(\frac{1}{x-2}\right)$
(4) $\displaystyle\lim_{x\to \infty} \frac{3-x}{x^2-3x+2}$
(5) $\displaystyle\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} e^x & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ \tan^2 x + 1 & \text{if } x > 0 \end{cases}$$

Solution.

(1)

$$\lim_{x\to -1} \frac{x^2-3x-4}{x+1} = \lim_{x\to -1} \frac{(x-4)(x+1)}{x+1} = \lim_{x\to -1}(x-4) = -5$$

(2)

$$\lim_{x\to \frac{1}{2}} \ln(\sin(\pi x)) = \ln(\sin(\pi \frac{1}{2})) = \ln 1 = 0$$

(3) Use squeeze theorem. Since $-1 \leqslant \sin\left(\frac{1}{x-2}\right) \leqslant 1$, we have

$$-(x^2-4)^2 \leqslant (x^2-4)^2 \sin\left(\frac{1}{x-2}\right) \leqslant (x^2-4)^2\,.$$

The limits $\lim_{x\to 2} -(x^2-4)^2 = \lim_{x\to 2}(x^2-4)^2 = 0$, thus

$$\lim_{x\to 2}(x^2-4)^2 \sin\left(\frac{1}{x-2}\right) = 0\,.$$

(4)

$$\lim_{x\to\infty} \frac{3-x}{x^2-3x+2} = \lim_{x\to\infty} \frac{\frac{3-x}{x^2}}{\frac{x^2-3x+2}{x^2}} = \lim_{x\to\infty} \frac{\frac{3}{x^2}-\frac{1}{x}}{1-\frac{3}{x}+\frac{2}{x^2}} = \frac{0-0}{1-0+0} = 0.$$

(5)

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = e^0 = 1\,;$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (\tan^2 x + 1) = \tan^2 0 + 1 = 1.$$

Because the left limit and the right limit are both equal to 1, $\lim_{x\to 0} f(x) = 1$.

4. Let $g(t) = \dfrac{t+3}{t-1}$.

(1) Find the equation(s) of all vertical asymptote(s) of g.
(2) Find the equation(s) of all horizontal asymptotes of g.
(3) Find $g^{-1}(t)$.

Solution.

(1) The only point where the function is not defined is $t = 1$. At this point we have

$$\lim_{t\to 1-} \frac{t+3}{t-1} = -\infty$$

since the numerator approaches 2 while the denominator approaches 0 from the left side, and

$$\lim_{t\to 1+} \frac{t+3}{t-1} = \infty$$

since the numerator approaches 2 while the denominator approaches 0 from the right side. Therefore $t = 1$ is the only vertical asymptote of g.

(2)

$$\lim_{t\to -\infty} g(t) = \lim_{t\to -\infty} \frac{t+3}{t-1} = \lim_{t\to -\infty} \frac{1+\frac{3}{t}}{1-\frac{1}{t}} = 1\,.$$

$$\lim_{t\to \infty} g(t) = \lim_{t\to \infty} \frac{t+3}{t-1} = \lim_{t\to \infty} \frac{1+\frac{3}{t}}{1-\frac{1}{t}} = 1.$$

The only horizontal asymptote of g is $y = 1$.

(3) If $y = \dfrac{t+3}{t-1}$, then

$$y(t-1) = t+3 \Rightarrow yt - y = t+3 \Rightarrow yt - t = y+3 \Rightarrow t(y-1) = y+3 \Rightarrow t = \frac{y+3}{y-1},$$

Therefore

$$g^{-1}(t) = \frac{t+3}{t-1}.$$

5. Show there exists a number a between $[0, \frac{\pi}{2}]$ such that the graph of $x^2 - \sin x$ has a horizontal tangent line at a.

Solution.

The graph of $x^2 - \sin x$ has a has a horizontal tangent line at a is equivalent to the derivative of $x^2 - \sin x$ is zero at a.

$(x^2 - \sin x)' = 2x - \cos x$. Let $2x - \cos x = f(x)$. It's a continuous function, in particular it's continuous on $[0, \frac{\pi}{2}]$.

$f(0) = 0 - 1 = -1 < 0$, $f(\frac{\pi}{2}) = \pi - 0 = \pi > 0$. By Intermediate Value Theorem there exists a number a between 0 and $\frac{\pi}{2}$ such that $f(a) = 0$. This a is the number desired.

6. Using the limit definition of the derivative, compute the derivative of $f(x) = 2\sqrt{x}$. What is the equation of the tangent line to the curve when $x = 1$?

Solution.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0} \frac{2\sqrt{x+h}-2\sqrt{x}}{h} = \lim_{h\to 0} \frac{2(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{2(x+h)-2x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{2h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{2}{\sqrt{x+h}+\sqrt{x}} = \frac{2}{\sqrt{x}+\sqrt{x}} \\ &= \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}} \end{aligned}$$

Plug $x = 1$ in $f'(x)$, we obtain $f'(1) = 1$. The tangent line passes through $(1, f(1)) = (1, 2)$ and has slope 1. Using the point-slope form, the equation of the tangent line is

$$y - 2 = 1(x - 1),$$

which simplifies to

$$y = x + 1.$$

7. Find the derivatives of the following functions:

(1) $f(x) = x^5 - x^{3/4} + 1$
(2) $f(x) = x \ln x$
(3) $f(x) = \sin(2e^x)$
(4) $f(x) = \dfrac{x^2-1}{x^2+1}$
(5) $f(x) = \ln\left(\dfrac{\sqrt{x}\cot x}{e^x}\right)$
(6) $f(x) = |x|$

Solution.

(1) $f'(x) = 5x^4 - \dfrac{3}{4}x^{-\frac{1}{4}}$
(2) $f'(x) = \ln x + x \cdot \dfrac{1}{x} = \ln x + 1$
(3) $f'(x) = \cos(2e^x) \cdot (2e^x)$
(4) $f'(x) = \dfrac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$
(5) $$\begin{aligned} f'(x) &= \frac{e^x}{\sqrt{x}\cot x} \cdot \frac{(\frac{1}{2}x^{-\frac{1}{2}}\cot x - \sqrt{x}\csc^2 x)e^x - \sqrt{x}\cot x \cdot e^x}{e^{2x}} \\ &= \frac{(\frac{1}{2}x^{-\frac{1}{2}}\cot x - \sqrt{x}\csc^2 x)e^x - \sqrt{x}\cot x \cdot e^x}{\sqrt{x}\cot x \cdot e^x} = \frac{\frac{1}{2}x^{-\frac{1}{2}}\cot x - \sqrt{x}\csc^2 x - \sqrt{x}\cot x}{\sqrt{x}\cot x} \end{aligned}$$

(6) First of all, write the given function as a piecewise defined function

$$f(x) = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

The derivative of $f(x)$ can be calculated in three cases. When $x > 0$, $f(x) = x$. Therefore its derivative is $f'(x) = 1$. When $x < 0$, $f(x) = -x$. Therefore its derivative is $f'(x) = -1$.

When $x = 0$, the situation is a little more complicated. On two sides of $x = 0$, the function is given by two different formulas. So we need to use the definition of limit to calculate the derivative at $x = 0$, that is, we need to calculate the limit

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}.$$

We have

$$\lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{h - 0}{h} = \lim_{h \to 0+} 1 = 1$$

and

$$\lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0-} \frac{-h - 0}{h} = \lim_{h \to 0-} -1 = -1.$$

The two one-sided limits do not agree, hence the limit

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist, which implies $f(x) = |x|$ is not differentiable at $x = 0$.

In summary, we have

$$f'(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2^t + t^3 + 1$ where t is measured in seconds.

(1) Find the average velocity of the particle during $[1, 3]$.
(2) Find the velocity of the particle at $t = 1$.
(3) Find the acceleration of the particle at $t = 1$.

Solution.

(1) We have the formula

$$\text{average velocity} = \frac{\text{distance travelled}}{\text{time elapsed}}.$$

Therefore the average velocity during the time period $[1, 3]$ is

$$\begin{aligned} v_{[1,3]} &= \frac{s(3) - s(1)}{3 - 1} = \frac{(2^3 + 3^3 + 1) - (2^1 + 1^3 + 1)}{3 - 1} \\ &= \frac{36 - 4}{3 - 1} = 16 \quad \text{(centimeters/second)}. \end{aligned}$$

(2) From the position function

$$s(t) = 2^t + t^3 + 1,$$

we can get the velocity function

$$v(t) = s'(t) = 2^t \ln 2 + 3t^2.$$

So the velocity at $t = 1$ is

$$v(1) = 2^1 \ln 2 + 3 \cdot 1^2 = 2 \ln 2 + 3 \quad \text{(centimeters/second)}.$$

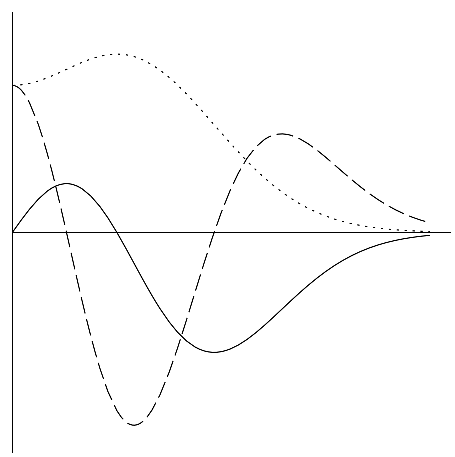
(3) Furthermore, the acceleration function is the derivative of the velocity.

$$a(t) = v'(t) = 2^t (\ln 2)^2 + 6t.$$

At $t = 1$, the acceleration is

$$a(1) = 2^1 (\ln 2)^2 + 6 \cdot 1 = 2(\ln 2)^2 + 6 \quad \text{(centimeters/second}^2\text{)}.$$

9. The figure shows the graphs of f, f' and f''. Identify each curve and explain your choices.



Solution.

We first look at the dotted curve. At one point on the curve, the tangent line is horizontal, which implies its derivative is 0 at this point. On the left of this point, the function is increasing, which implies its derivative is positive. Similarly, on the right of this point, the function is decreasing, which implies its derivative is negative. It seems that the solid curve satisfies all these requirements. So it is the candidate of the derivative function of the dotted curve.

If we further look at the solid curve, it has horizontal tangents at two points. The function has negative derivative on the interval between these two points, and positive derivative outside this interval. So the dashed curve looks like the derivative of the solid curve.

Finally, the dashed curve have two horizontal tangents. However, neither of the other two functions has zero value at both of these points. Therefore the derivative function of the dashed curve doesn't appear in the figure.

In summary, the solid curve is the derivative of the dotted curve, and the dashed curve is the derivative of the solid curve. So the dotted curve is f, the solid curve is f', and the dashed curve is f''.

10. Sketch a possible graph of $f(x)$ which satisfies all the conditions:

(i) $f(0) = 1$, (ii) $\lim_{x\to-\infty} f(x) = 0$, (iii) $f'(0) = 1$, (iv) f is increasing on $[-1, 1]$,
(v) f is concave downward on $(-\infty, -1)$, (vi) f is concave upward on $(0, 1]$,
(vii) $\lim_{x\to 3^-} f(x) = 5$, (viii) $\lim_{x\to 3^+} f(x) = 2$, (ix) f is decreasing on $[3, \infty)$,
(x) $\lim_{x\to\infty} f(x) = -\infty$.

Solution.

Note that there are many different correct solutions. The following is a possible graph.