Midterm Exam Solutions

1. *(10 points)* Circle **True** or **False** for each of the following statement. You do not need to explain your answers.

True **False** The function $g(x) = |x|$ is continuous on $(-\infty, \infty)$.

True **False** If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x)g(x)$ does not exist.

True **False** If $f(x)$ is one-to-one and 1 is in the domain of $f(x)$, then $f^{-1}(f(1)) = 1$.

True **False** If $f(x)$ is not defined at $x = 1$, then $\lim_{x\to 1} f(x)$ does not exist.

True **False** If $f(x)$ is discontinuous at $x = 1$, then $f(x)$ is not differentiable at $x = 1$.

Solution.

(a) True.

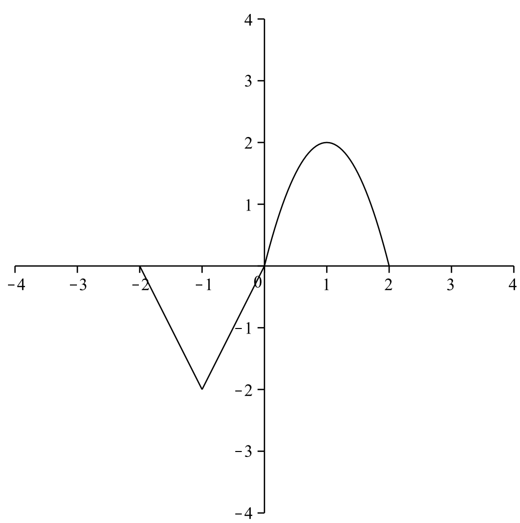
(b) False. A counterexample is $f(x) = 0$ (or x^2) and $g(x) = \sin\dfrac{1}{x}$.

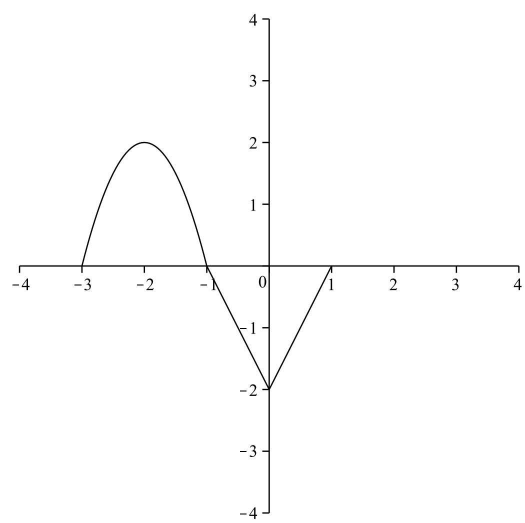
(c) True.

(d) False. For the limit to exist, the function doesn't have to be defined at that point.

(e) True.

2. *(4 points)* The graph of $y = f(x)$ is given as below. Sketch the graph of $y = f(-x-1)$ below. No explanation is necessary.





3. *(18 points)* Find each of the following limits, with complete justification. If there is an infinite limit, then explain whether it is ∞ or $-\infty$.

(a) $\lim_{x\to 0} \frac{x}{(1+x)^2-(1-x)^2}$

$$\begin{aligned}\lim_{x\to 0} \frac{x}{(1+x)^2-(1-x)^2} &= \lim_{x\to 0} \frac{x}{(1+2x+x^2)-(1-2x+x^2)} \\ &= \lim_{x\to 0} \frac{x}{4x} = \lim_{x\to 0} \frac{1}{4} = \frac{1}{4}\end{aligned}$$

(b) $\lim_{x\to\infty} \frac{\sqrt{4x^2+6}}{3-2x}$

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{4x^2+6}}{3-2x} &= \lim_{x\to\infty} \frac{\frac{\sqrt{4x^2+6}}{x}}{\frac{3-2x}{x}} \\ &= \lim_{x\to\infty} \frac{\sqrt{4+\frac{6}{x^2}}}{\frac{3}{x}-2} \\ &= \frac{\sqrt{4+0}}{0-2} = -1\end{aligned}$$

(c) $\lim_{x\to 2} \frac{3}{(2-x)^3}$

$\lim_{x\to 2^-} \frac{3}{(2-x)^3} = \infty$ since the numerator is a finite positive number while the denominator approaches 0 from the positive side. $\lim_{x\to 2^+} \frac{3}{(2-x)^3} = -\infty$ since the numerator is a finite positive number while the denominator approaches 0 from the negative side. Therefore the limit doesn't exist since the left side limit and the right side limit do not agree.

(d) $\lim_{t\to 0} e^{-t}\sin(2\pi t)$

$$\lim_{t\to 0} e^{-t}\sin(2\pi t) = e^0 \sin(2\pi\cdot 0) = 0$$

4. *(8 points)*

(a) Complete the following definition in precise terms: a function $f(x)$ is said to be *continuous* at the point $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

(b) Let

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Is $f(x)$ continuous at $x = 0$? Give a rigorous justification for your answer.

Solution.
We need to find $\lim_{x \to 0} f(x)$ and $f(0)$.
We have

$$-1 \leqslant \sin \frac{1}{x} \leqslant 1.$$

Since $x^4 \geqslant 0$, we can multiply each function by x^4 and get

$$-x^4 \leqslant x^4 \sin \frac{1}{x} \leqslant x^4.$$

It's easy to see that

$$\lim_{x \to 0} x^4 = \lim_{x \to 0} (-x^4) = 0.$$

By Squeeze Theorem, we have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^4 \sin \frac{1}{x} = 0.$$

We also have $f(0) = 0$. So

$$\lim_{x \to 0} f(x) = f(0),$$

which implies that $f(x)$ is continuous at $x = 0$.

5. *(8 points)* Prove that the equation $e^x = 4 - 3x$ has a root in the interval $[0, 1]$. Show all your reasoning.

Solution.
Let
$$f(x) = e^x - (4 - 3x) = e^x - 4 + 3x.$$
$f(x)$ is a continuous function. In particular, it is continuous on $[0, 1]$. We have
$$f(0) = e^0 - 4 + 0 = -3 < 0,$$
and
$$f(1) = e^1 - 4 + 3 = e - 1 > 0.$$
By Intermediate Value Theorem, there exists a number a between 0 and 1, such that $f(a) = 0$. Such a number a is a root of the original equation, as desired.

6. *(16 points)* Let $f(x) = \dfrac{8}{3 + e^x}$.

(a) Find, with complete mathematical justification, the equation(s) of all *vertical* asymptote(s) of f, or explain why none exists.

Solution.
f doesn't have any vertical asymptote. For any x, the denominator $3 + e^x > 0$, which implies the domain of f is all real numbers. Furthermore, since f is an elementary function, it is continuous on its domain. Therefore f is continuous on $(-\infty, \infty)$. It doesn't have any discontinuity. In particular, it doesn't have any infinite discontinuity.

(b) Find, with complete mathematical justification, the equation(s) of all *horizontal* asymptote(s) of f, or explain why none exists.

Solution.
We only need to calculate the limit of f at ∞ and $-\infty$. We have
$$\lim_{x \to \infty} \frac{8}{3 + e^x} = 0$$

since the numerator is constant while the denominator approaches ∞. We also have

$$\lim_{x\to-\infty}\frac{8}{3+e^x}=\frac{8}{3+0}=\frac{8}{3}$$

since $\lim_{x\to-\infty} e^x = 0$. Therefore the function f has two horizontal asymptotes, which are $y = 0$ and $y = \frac{8}{3}$.

(c) It is a fact that $f(x)$ is a one-to-one function. Find an expression for its inverse function $f^{-1}(x)$. Show your computation.

Solution.
Let

$$y=\frac{8}{3+e^x}.$$

Interchanging x and y, we get

$$x=\frac{8}{3+e^y}.$$

Solve for y in terms of x, we have

$$\begin{aligned}3x+xe^y&=8,\\ e^y&=\frac{8-3x}{x},\\ y&=\ln\frac{8-3x}{x}.\end{aligned}$$

(d) Find the domain of $f^{-1}(x)$. Show your computation.

Solution.
The only restriction to x is

$$\frac{8-3x}{x}>0.$$

Since the function $\frac{8-3x}{x}$ might change its sign at $x = 0$ and $x = \frac{8}{3}$, we use the two points to separate the number line into three intervals. On $(-\infty, 0)$, $8 - 3x > 0$ and $x < 0$, so $\frac{8-3x}{x} < 0$; on $(0, \frac{8}{3})$, $8 - 3x > 0$ and $x > 0$, so $\frac{8-3x}{x} > 0$; on $(\frac{8}{3}, \infty)$, $8 - 3x < 0$ and $x > 0$, so $\frac{8-3x}{x} < 0$. Therefore, the domain of this function is $(0, \frac{8}{3})$.

7. *(12 points)* Let $f(x) = \sqrt{x^2+1}$.

(a) Find a formula for $f'(x)$ using the limit definition of the derivative. Show all steps of your computation.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{(x+h)^2+1}-\sqrt{x^2+1})(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} \\
&= \lim_{h\to 0} \frac{((x+h)^2+1)-(x^2+1)}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \lim_{h\to 0} \frac{2xh+h^2}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} \\
&= \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}} = \frac{2x+0}{\sqrt{(x+0)^2+1}+\sqrt{x^2+1}} \\
&= \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}
\end{aligned}
$$

(b) Use your answer in part (a) to find the equation of the tangent line to the graph of $f(x)$ at $x = 1$.

Solution.
From the formula that we computed above, the slope of the tangent line at $x = 1$ is

$$f'(1) = \frac{1}{\sqrt{2}}.$$

Furthermore, when $x = 1$, $f(1) = \sqrt{2}$. So the equation of the line is

$$y - \sqrt{2} = \frac{1}{\sqrt{2}}(x-1).$$

(c) Suppose the function $f(x)$ represents the position of a particle which moves along a straight line at time x. Explain, in words, the practical meaning of the slope of the tangent line in part (b).

Solution.
It's the instantaneous velocity of the particle at the time $x = 1$. In other words, it's the infinitesimal rate of change of the position of the particle at the time $x = 1$.

8. *(8 points)* Find the derivatives of the following functions, using any method you like. You do not need to simplify your answers.

(a) $f(x) = \dfrac{2x}{\sqrt{x^2+1}}$

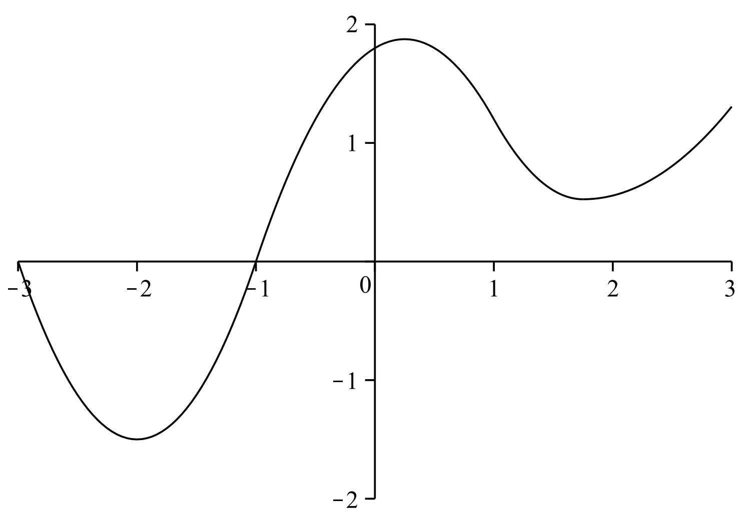
$$f'(x) = \frac{2\sqrt{x^2+1} - 2x \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x}{x^2+1}$$

(b) $g(x) = e^{-x}\cos x$

$$f'(x) = e^{-x} \cdot (-1)\cos x + e^{-x}(-\sin x)$$

9. *(6 points)* The graph of $f(x)$ is given as below. List the following six quantities in increasing order (from smallest to largest). No justification is necessary.

$f(0)$ $f'(-1)$ $f'(2)$ $f''(0)$ $f''(1)$ The number 1



Solution.

$$f''(0) < f''(1) < f'(2) < 1 < f(0) < f'(-1)$$

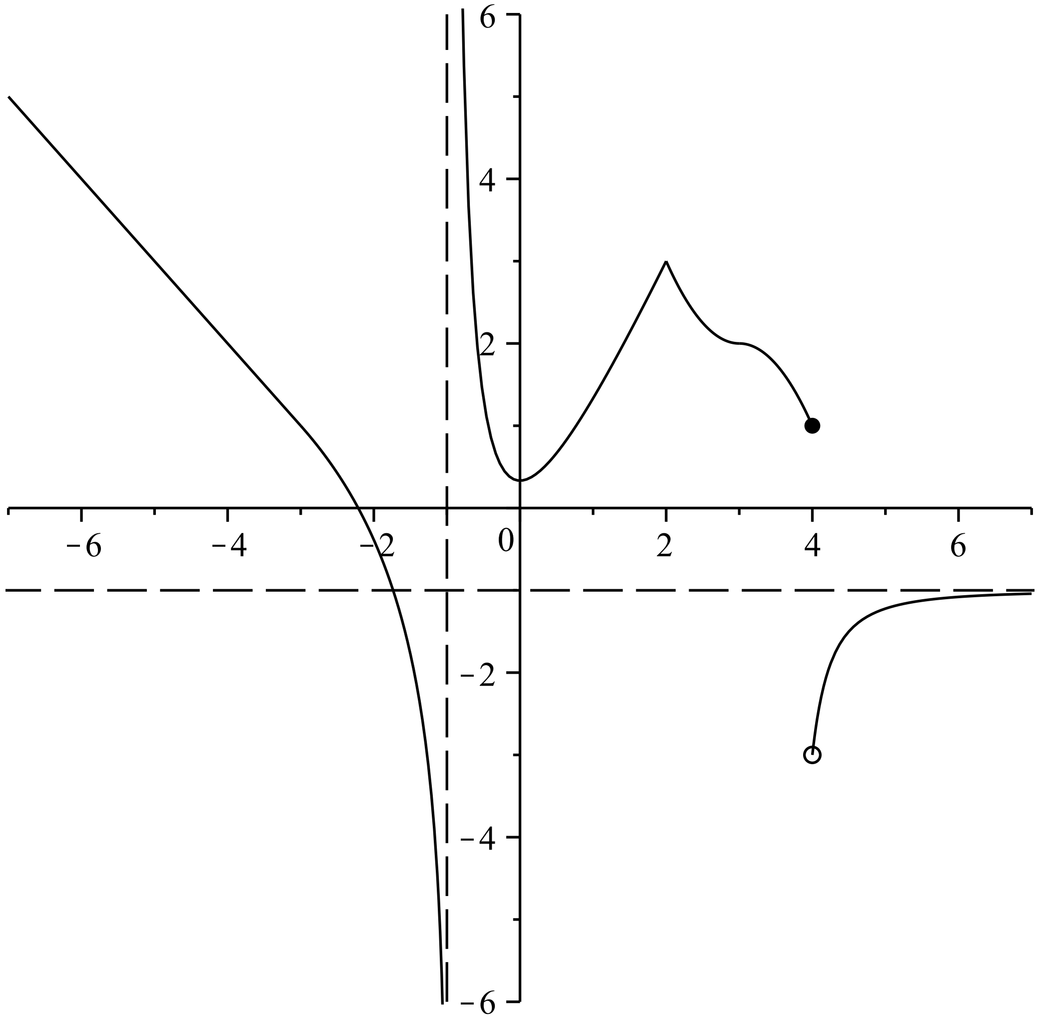
Reason: $f''(0)$ is the only one which is significantly negative since the function is concave downward there. At $x = 1$, the function changes from being concave down to being concave up, hence $f''(1) = 0$. At $x = 2$, the angle of elevation of the tangent line is between 0 and 45°, so the slope $f'(2)$ is between 0 and 1. $f(0)$ is obviously between 1 and 2. At $x = -1$, the slope of the tangent line is very large (in fact, if you sketch the tangent line and estimate its slope, you will find it's roughly equal to 3).

10. *(10 points)* Sketch the graph of a function $f(x)$ with all of the following properties. Be sure to label the scales of your axes appropriately. No explanation is necessary.

- The domain of f is all real numbers except $x = -1$;
- f is continuous on its entire domain except $x = 4$;
- f is not differentiable at $x = 2$;
- f has asymptotes $x = -1$ and $y = -1$;
- $f'(x) = -1$ for $x < -3$;
- f is concave upward for $-1 < x < 2$;
- f has a horizontal tangent at $x = 0$;
- f has an inflection point at $(3, 2)$;
- f is increasing for $x > 4$.

Solution.

There are many correct answers. The following graph is one of the many possibilities.


6
4
2
0
−2
−4
−6
−6
−4
−2
2
4
6