Math 19: Calculus **Summer 2009**

Homework 3 - Solutions

Homework scores are out of 30 points. Ten problems were graded, each worth 3 points. The graded problems from this homework set are 6, 22, and 44 from section 2.6, 10, 26, and 38 from section 2.7, and 2, 12, 16, and 28 from section 2.8.

Please check that your solutions are correct on the ungraded problems.

Section 2.6

6.

The slope m of the tangent line is given by

$$m = \lim_{x\to 2} \frac{x^3 - 3x + 1 - 3}{x - 2} = \lim_{x\to 2} \frac{x^3 - 3x - 2}{x - 2} = \lim_{x\to 2} \frac{(x^2 + 2x + 1)(x - 2)}{x - 2}$$

You can use polynomial division to get the last term above.

$$\lim_{x\to 2} \frac{(x^2 + 2x + 1)(x - 2)}{x - 2} = \lim_{x\to 2} x^2 + 2x + 1 = 4 + 4 + 1 = 9$$

The equation of the tangent line is then $y - 3 = 9(x - 2)$.

10.

(a) Using (1),

$$m = \lim_{x\to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a} = \lim_{x\to a} \frac{\frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}}}{x - a} = \lim_{x\to a} \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{ax}\,(x - a)(\sqrt{a} + \sqrt{x})}$$

$$= \lim_{x\to a} \frac{a - x}{\sqrt{ax}\,(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x\to a} \frac{-1}{\sqrt{ax}\,(\sqrt{a} + \sqrt{x})} = \frac{-1}{\sqrt{a^2}\,(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2}$$

(b) At $(1, 1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 1) \Leftrightarrow y = -\frac{1}{2}x + \frac{3}{2}$.

At $(4, \frac{1}{2})$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x - 4) \Leftrightarrow y = -\frac{1}{16}x + \frac{3}{4}$.

14.

a) The velocity after 1 second is the slope of the tangent line to the position function at $t = 1$:

$$v(1) = \lim_{h\to 0} \frac{H(1 + h) - H(1)}{h} = \lim_{h\to 0} \frac{10(1 + h) - 1.86(1 + h)^2 - (10 - 1.86(1)^2)}{h}$$

$$= \lim_{h\to 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 10 + 1.86}{h} = \lim_{h\to 0} \frac{10h - 1.86 - 3.72h - 1.86h^2 + 1.86}{h}$$

$$= \lim_{h\to 0} \frac{h(10 - 3.72 - 1.86h)}{h} = \lim_{h\to 0} \frac{h(10 - 3.72 - 1.86h)}{h}$$

$$= \lim_{h\to 0}(10 - 3.72 - 1.86h) = 10 - 3.72 = 6.28 \text{ m/s}$$

b)

$$v(a) = \lim_{h\to 0} \frac{H(a+h)-H(a)}{h} = \lim_{h\to 0} \frac{10(a+h)-1.86(a+h)^2-(10a-1.86(a)^2)}{h}$$
$$= \lim_{h\to 0} \frac{10a+10h-1.86a-3.72ah-1.86h^2-10a+1.86a}{h} = \lim_{h\to 0} \frac{10h-3.72ah-1.86h^2}{h}$$
$$= \lim_{h\to 0} 10-3.72a-1.86h = 10-3.72a \text{ m/s}$$

c)

The rock will hit the surface when the height is 0.
That is $10t - 1.86t^2 = 0 \iff t(10-1.86t) = 0 \iff t = \frac{10}{1.86}$ (since t can't equal 0)

d)

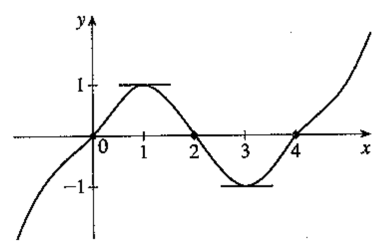
Using the time from part c),
$$v(\frac{10}{1.86} = 10 - 3.72(\frac{10}{1.86}) = 10 - 2(10) = -10 \text{ m/s}$$
Thus the arrow will have a velocity of -10 m/s.

20.

The problem tells us that the point $(4,3)$ is on $y = f(x)$ so $f(4) = 3$.
$f'(4)$ is the slope of the tangent line at the point $x = 4$ which we can determine given two points it passes through:
$$f'(4) = \frac{2-3}{0-4} = \frac{-1}{-4} = \frac{1}{4}$$

22.

We begin by drawing a curve through the origin with a slope of 1 to satisfy $g(0) = 0$ and $g'(0) = 1$. We round off our figure at $x = 1$ to satisfy $g'(1) = 0$, and then pass through $(2,0)$ with slope -1 to satisfy $g(2) = 0$ and $g'(2) = -1$. We round the figure at $x = 3$ to satisfy $g'(3) = 0$, and then pass through $(4,0)$ with slope 1 to satisfy $g(4) = 0$ and $g'(4) = 1$. Finally we extend the curve on both ends to satisfy $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to-\infty} g(x) = -\infty$.



44.

a) i) $$avg_{2005,2007} = \frac{N(2007)-N(2005)}{2007-2005} = \frac{15011-10241}{2} = \frac{4770}{2} = 2385 \text{ locations/year}$$

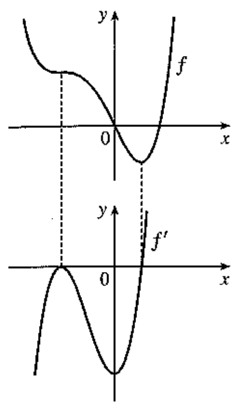
ii) $$avg_{2005,2006} = \frac{N(2006)-N(2005)}{2006-2005} = \frac{12440-10241}{1} = 2199 \text{ locations/year}$$

iii) $$avg_{2004,2005} = \frac{N(2005)-N(2004)}{2005-2004} = \frac{10241-8569}{1} = 1672 \text{ locations/year}$$

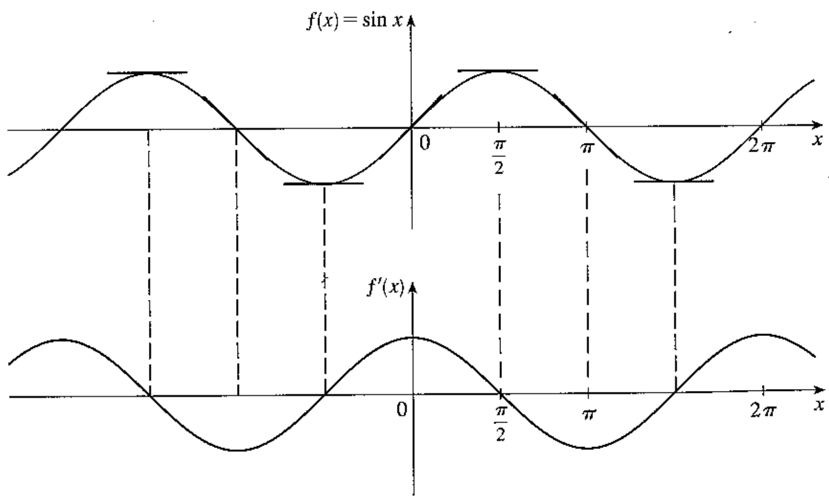
b) Using the values from *ii*) and *iii*) we have $\frac{2199+1672}{2} = 1935.5$ locations/year.

Section 2.7

10.



14.



26.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)^2-1}{2(x+h)-3} - \frac{x^2-1}{2x-3}}{h} = \lim_{h\to 0} \frac{\frac{x^2+2xh+h^2-1}{2x+2h-3} - \frac{x^2-1}{2x-3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(x^2+2xh+h^2-1)(2x-3)-(x^2-1)(2x+2h-3)}{(2x+2h-3)(2x-3)}}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2-1)(2x-3)-(x^2-1)(2x+2h-3)}{(2x+2h-3)(2x-3)h}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2-1)(2x-3)+(-x^2+1)(2x+2h-3)}{(2x+2h-3)(2x-3)h}$$

$$= \lim_{h\to 0} \frac{2x^3+4x^2h+2xh^2-2x-3x^2-6xh-3h^2+3-2x^3-2x^2h+3x^2+2x+2h-3}{(2x+2h-3)(2x-3)h}$$

$$= \lim_{h\to 0} \frac{2x^2h+2xh^2-6xh-3h^2+2h}{(2x+2h-3)(2x-3)h}$$

$$= \lim_{h \to 0} \frac{2x^2 + 2xh - 6x - 3h + 2}{(2x + 2h - 3)(2x - 3)}$$
$$= \frac{2x^2 - 6x + 2}{(2x - 3)(2x - 3)}$$

The domain of f and f' is all real numbers except $x = 3/2$.

38.

f is not differentiable at $x = -1$, because there is a discontinuity there, and at $x = 2$, because the graph has a corner there.

44.

We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

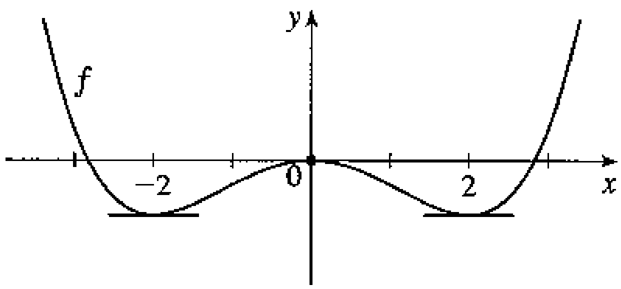
Section 2.8

2.

a) f is increasing when $f'(x) > 0$ so on $(-2, 0) \cup (2, \infty)$
f is idecreasing when $f'(x) < 0$ so on $(-\infty, -2) \cup (0, 2)$

b) f has a local maximum where f' changes from positive to negative, so at $x = 0$. Similarly, it has a local minimum where f' changes from negative to positive, so at $x = -2$ and $x = 2$

c)



6.

First we need to determine which graph is f and which is f'. The maxima and minima of both graphs correspond to zeros of the other graph so we need to use further clues to determine which is which. Notice that the when the red graph is increasing the blue graph is positive. Similarly when the red graph is decreasing, the blue graph is negative. Since we don't see the same behavior from the blue to red graph, we can now say that the red graph is the graph of f and the blue graph is the graph of f'.

We can see from the graph that $f'(-1)$ is a positive number. Since f'' is the derivative of f', $f''(-1)$ is the slope of the line tangent to f' at $x = -1$. Since the blue graph is decreasing at $x = -1$, we know that $f''(-1)$ must be negative. Therefore, $f'(-1) > f''(-1)$

12.

(a) If the position function is increasing, then the particle is moving toward the right. This occurs on t-intervals $(0, 2)$ and $(4, 6)$. If the function is decreasing, then the particle is moving toward the left—that is, on $(2, 4)$.

(b) The acceleration is the second derivative and is positive where the curve is concave upward. This occurs on $(3, 6)$. The acceleration is negative where the curve is concave downward—that is, on $(0, 3)$.

16.

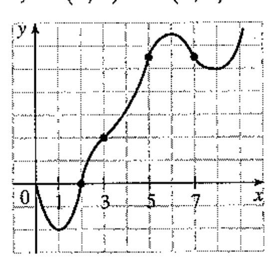
(a) f is increasing where f' is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where f' is negative, on $(0, 1)$ and $(6, 8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where f' is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

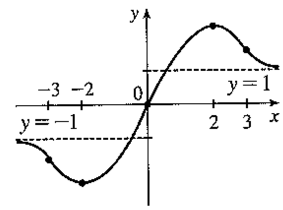
(d) There are points of inflection where f changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e)



24.

$f'(x) > 0$ if $|x| < 2 \;\Rightarrow\; f$ is increasing on $(-2, 2)$. $f'(x) < 0$ if $|x| > 2 \;\Rightarrow\; f$ is decreasing on $(-\infty, -2)$ and $(2, \infty)$. $f'(2) = 0$, so f has a horizontal tangent (and local maximum) at $x = 2$. $\lim_{x\to\infty} f(x) = 1 \;\Rightarrow\; y = 1$ is a horizontal asymptote. $f(-x) = -f(x)$ $\Rightarrow\; f$ is an odd function (its graph is symmetric about the origin). Finally, $f''(x) < 0$ if $0 < x < 3$ and $f''(x) > 0$ if $x > 3$, so f is CD on $(0, 3)$ and CU on $(3, \infty)$.



28.

(a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{[(x+h)^4 - 2(x+h)^2] - (x^4 - 2x^2)}{h}$

$= \lim_{h\to 0} \dfrac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2x^2 - 4xh - 2h^2) - (x^4 - 2x^2)}{h}$

$= \lim_{h\to 0} \dfrac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 4xh - 2h^2}{h} = \lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 - 4x - 2h) = 4x^3 - 4x$

$f''(x) = \lim_{h\to 0} \dfrac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \dfrac{[4(x+h)^3 - 4(x+h)] - (4x^3 - 4x)}{h}$

$= \lim_{h\to 0} \dfrac{(4x^3 + 12x^2h + 12xh^2 + 4h^3 - 4x - 4h) - (4x^3 - 4x)}{h} = \lim_{h\to 0} \dfrac{12x^2h + 12xh^2 + 4h^3 - 4h}{h}$

$= \lim_{h\to 0} (12x^2 + 12xh + 4h^2 - 4) = 12x^2 - 4$

(b) $f'(x) > 0 \;\Leftrightarrow\; 4x^3 - 4x > 0 \;\Leftrightarrow\; 4x(x^2 - 1) > 0 \;\Leftrightarrow\; 4x(x+1)(x-1) > 0$, so f is increasing on $(-1, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -1)$ and $(0, 1)$.

(c) $f''(x) > 0 \;\Leftrightarrow\; 12x^2 - 4 > 0 \;\Leftrightarrow\; 12x^2 > 4 \;\Leftrightarrow\; x^2 > \frac{1}{3} \;\Leftrightarrow\; |x| > \sqrt{\frac{1}{3}}$, so f is CU on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$ and $\left(\sqrt{\frac{1}{3}}, \infty\right)$ and f is CD on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$.