Math 19: Calculus **Summer 2009**

Homework 4 - Solutions

Homework scores are out of 30 points.

Please check that your solutions are correct on the ungraded problems.

Section 3.1

4.

$f(x) = \sqrt{30}$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

24.

$$v = \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2 = \left(\sqrt{x}\right)^2 + 2\sqrt{x}\cdot\frac{1}{\sqrt[3]{x}} + \left(\frac{1}{\sqrt[3]{x}}\right)^2 = x + 2x^{1/2-1/3} + 1/x^{2/3} = x + 2x^{1/6} + x^{-2/3} \quad \Rightarrow$$

$$v' = 1 + 2\left(\tfrac{1}{6}x^{-5/6}\right) - \tfrac{2}{3}x^{-5/3} = 1 + \tfrac{1}{3}x^{-5/6} - \tfrac{2}{3}x^{-5/3} \quad \text{or} \quad 1 + \frac{1}{3\sqrt[6]{x^5}} - \frac{2}{3\sqrt[3]{x^5}}$$

46 (a) (b).

(a) $s = t^4 - 2t^3 + t^2 - t \quad \Rightarrow$

$v(t) = s'(t) = 4t^3 - 6t^2 + 2t - 1 \quad \Rightarrow$

$a(t) = v'(t) = 12t^2 - 12t + 2$

(b) $a(1) = 12(1)^2 - 12(1) + 2 = 2\,\text{m/s}^2$

48.

$f(x) = x^3 - 4x^2 + 5x \quad \Rightarrow \quad f'(x) = 3x^2 - 8x + 5 \quad \Rightarrow \quad f''(x) = 6x - 8.$

$f''(x) > 0 \quad \Rightarrow \quad 6x - 8 > 0 \quad \Rightarrow \quad x > \frac{4}{3}$. f is concave upward when $f''(x) > 0$; that is, on $\left(\frac{4}{3}, \infty\right)$.

52.

$y = x\sqrt{x} = x^{3/2} \quad \Rightarrow \quad y' = \frac{3}{2}x^{1/2}$. The slope of the line $y = 1 + 3x$ is 3, so the slope of any line parallel to it is also 3. Thus, $y' = 3 \quad \Rightarrow \quad \frac{3}{2}x^{1/2} = 3 \quad \Rightarrow \quad \sqrt{x} = 2 \quad \Rightarrow \quad x = 4$, which is the x-coordinate of the point on the curve at which the slope is 3. The y-coordinate is $y = 4\sqrt{4} = 8$, so an equation of the tangent line is $y - 8 = 3(x - 4)$ or $y = 3x - 4$.

62.

$y = Ax^2 + Bx + C \;\Rightarrow\; y' = 2Ax + B \;\Rightarrow\; y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$
$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$
$$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (0)x + (0)$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \;\Rightarrow\; A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \;\Rightarrow\; A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \;\Rightarrow\; -1 - \frac{1}{2} - 2C = 0 \;\Rightarrow\; C = -\frac{3}{4}$.

68.

The slope of the curve $y = c\sqrt{x}$ is $y' = \dfrac{c}{2\sqrt{x}}$ and the slope of the tangent line $y = \frac{3}{2}x + 6$ is $\frac{3}{2}$. These must be equal at the point of tangency $\left(a, c\sqrt{a}\right)$, so $\dfrac{c}{2\sqrt{a}} = \dfrac{3}{2} \;\Rightarrow\; c = 3\sqrt{a}$. The y-coordinates must be equal at $x = a$, so $c\sqrt{a} = \frac{3}{2}a + 6 \;\Rightarrow\; \left(3\sqrt{a}\right)\sqrt{a} = \frac{3}{2}a + 6 \;\Rightarrow\; 3a = \frac{3}{2}a + 6 \;\Rightarrow\; \frac{3}{2}a = 6 \;\Rightarrow\; a = 4$. Since $c = 3\sqrt{a}$, we have $c = 3\sqrt{4} = 6$.

Section 3.2

2.

Quotient Rule: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = \dfrac{x^4 - 5x^3 + x^{1/2}}{x^2} \;\Rightarrow$

$$F'(x) = \frac{x^2(4x^3 - 15x^2 + \frac{1}{2}x^{-1/2}) - (x^4 - 5x^3 + x^{1/2})(2x)}{(x^2)^2} = \frac{4x^5 - 15x^4 + \frac{1}{2}x^{3/2} - 2x^5 + 10x^4 - 2x^{3/2}}{x^4}$$
$$= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \tfrac{3}{2}x^{-5/2}$$

Simplifying first: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = x^2 - 5x + x^{-3/2} \;\Rightarrow\; F'(x) = 2x - 5 - \frac{3}{2}x^{-5/2}$ (equivalent).

For this problem, simplifying first seems to be the better method.

4.

By the Product Rule, $g(x) = \sqrt{x}\,e^x = x^{1/2}e^x \;\Rightarrow\; g'(x) = x^{1/2}(e^x) + e^x\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}x^{-1/2}e^x(2x + 1)$.

8.

$$f(t) = \frac{2t}{4+t^2} \overset{\text{QR}}{\Rightarrow} f'(t) = \frac{(4+t^2)(2)-(2t)(2t)}{(4+t^2)^2} = \frac{8+2t^2-4t^2}{(4+t^2)^2} = \frac{8-2t^2}{(4+t^2)^2}$$

22.

$$f(x) = \frac{1-xe^x}{x+e^x} \overset{\text{QR}}{\Rightarrow} f'(x) = \frac{(x+e^x)(-xe^x)'-(1-xe^x)(1+e^x)}{(x+e^x)^2}$$

$$\overset{\text{PR}}{\Rightarrow} f'(x) = \frac{(x+e^x)[-(xe^x+e^x\cdot 1)]-(1+e^x-xe^x-xe^{2x})}{(x+e^x)^2}$$

$$= \frac{-x^2e^x-xe^x-xe^{2x}-e^{2x}-1-e^x+xe^x+xe^{2x}}{(x+e^x)^2} = \frac{-x^2e^x-e^{2x}-e^x-1}{(x+e^x)^2}$$

40.

$$g(x) = \frac{x}{e^x} \Rightarrow g'(x) = \frac{e^x\cdot 1 - x\cdot e^x}{(e^x)^2} = \frac{e^x(1-x)}{(e^x)^2} = \frac{1-x}{e^x} \Rightarrow$$

$$g''(x) = \frac{e^x\cdot(-1)-(1-x)e^x}{(e^x)^2} = \frac{e^x[-1-(1-x)]}{(e^x)^2} = \frac{x-2}{e^x} \Rightarrow$$

$$g'''(x) = \frac{e^x\cdot 1-(x-2)e^x}{(e^x)^2} = \frac{e^x[1-(x-2)]}{(e^x)^2} = \frac{3-x}{e^x} \Rightarrow$$

$$g^{(4)}(x) = \frac{e^x\cdot(-1)-(3-x)e^x}{(e^x)^2} = \frac{e^x[-1-(3-x)]}{(e^x)^2} = \frac{x-4}{e^x}.$$

The pattern suggests that $g^{(n)}(x) = \dfrac{(x-n)(-1)^n}{e^x}$. (We could use mathematical induction to prove this formula.)

42.

We are given that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$.

(a) $h(x) = 5f(x) - 4g(x) \Rightarrow h'(x) = 5f'(x) - 4g'(x)$, so

$h'(2) = 5f'(2) - 4g'(2) = 5(-2) - 4(7) = -10 - 28 = -38.$

(b) $h(x) = f(x)g(x) \Rightarrow h'(x) = f(x)g'(x) + g(x)f'(x)$, so

$h'(2) = f(2)g'(2) + g(2)f'(2) = (-3)(7) + (4)(-2) = -21 - 8 = -29.$

(c) $h(x) = \dfrac{f(x)}{g(x)} \Rightarrow h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, so

$$h'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{4(-2)-(-3)(7)}{4^2} = \frac{-8+21}{16} = \frac{13}{16}.$$

(d) $h(x) = \dfrac{g(x)}{1+f(x)} \Rightarrow h'(x) = \dfrac{[1+f(x)]\,g'(x) - g(x)\,f'(x)}{[1+f(x)]^2}$, so

$$h'(2) = \frac{[1+f(2)]\,g'(2) - g(2)\,f'(2)}{[1+f(x)]^2} = \frac{[1+(-3)](7)-4(-2)}{[1+(-3)]^2} = \frac{-14+8}{(-2)^2} = \frac{-6}{4} = -\frac{3}{2}.$$

50.

(a) $f(20) = 10{,}000$ means that when the price of the fabric is \$20/yard, 10,000 yards will be sold.

$f'(20) = -350$ means that as the price of the fabric increases past \$20/yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p) = pf(p) \;\Rightarrow\; R'(p) = pf'(p) + f(p)\cdot 1 \;\Rightarrow\; R'(20) = 20f'(20) + f(20)\cdot 1 = 20(-350) + 10{,}000 = 3000.$

This means that as the price of the fabric increases past \$20/yard, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/yard) due to selling less fabric, but this loss is more than made up for by the additional revenue due to the increase in price.

Section 3.3

2.

$$y = 2\csc x + 5\cos x \quad\Rightarrow\quad y' = -2\csc x\,\cot x - 5\sin x$$

8.

$$f(t) = \frac{\cot t}{e^t} \quad\Rightarrow\quad f'(t) = \frac{e^t(-\csc^2 t) - (\cot t)e^t}{(e^t)^2} = \frac{e^t(-\csc^2 t - \cot t)}{(e^t)^2} = -\frac{\csc^2 t + \cot t}{e^t}$$

12.

$$y = \frac{1 - \sec x}{\tan x} \quad\Rightarrow$$

$$y' = \frac{\tan x\,(-\sec x\,\tan x) - (1 - \sec x)(\sec^2 x)}{(\tan x)^2} = \frac{\sec x\left(-\tan^2 x - \sec x + \sec^2 x\right)}{\tan^2 x} = \frac{\sec x\,(1 - \sec x)}{\tan^2 x}$$

16.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x\,\tan x$$

22.

$y = \dfrac{1}{\sin x + \cos x} \;\Rightarrow\; y' = -\dfrac{\cos x - \sin x}{(\sin x + \cos x)^2}$ [Reciprocal Rule]. At $(0, 1)$, $y' = -\dfrac{1-0}{(0+1)^2} = -1$, and an equation of the tangent line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

36 a-e.

(a) $s(t) = 2\cos t + 3\sin t \quad \Rightarrow \quad v(t) = -2\sin t + 3\cos t \quad \Rightarrow$ (b)

$a(t) = -2\cos t - 3\sin t$

(c) $s = 0 \quad \Rightarrow \quad t_2 \approx 2.55.$ So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

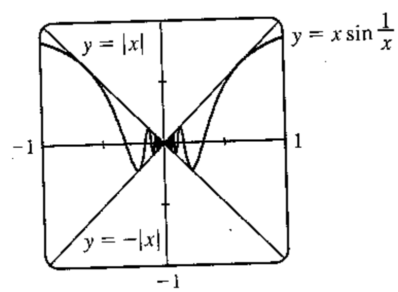
(d) $v = 0 \quad \Rightarrow \quad t_1 \approx 0.98,\ s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$, that is, when $t = t_2 + n\pi$, n a positive integer.

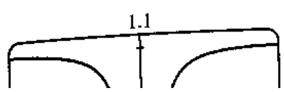
46 a-b.

(a) Let $\theta = \frac{1}{x}$. Then as $x \to \infty$, $\theta \to 0$, and $\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{\theta\to 0}\frac{1}{\theta}\sin\theta = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$

(b) Since $-1 \le \sin(1/x) \le 1$, we have (as illustrated in the figure) $-|x| \le x\sin(1/x) \le |x|$. We know that $\lim_{x\to 0}(|x|) = 0$ and $\lim_{x\to 0}(-|x|) = 0$; so by the Squeeze Theorem, $\lim_{x\to 0} x\sin(1/x) = 0.$



(c)



Section 3.4

2.

Let $u = g(x) = 2x^3 + 5$ and $y = f(u) = u^4$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (4u^3)(6x^2) = 24x^2(2x^3+5)^3.$

6.

Let $u = g(x) = 2 - e^x$ and $y = f(u) = \sqrt{u}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\frac{1}{2}u^{-1/2})(-e^x) = -\frac{e^x}{2\sqrt{2-e^x}}.$

18.

$y = e^{-2t}\cos 4t \quad \Rightarrow \quad y' = e^{-2t}(-\sin 4t \cdot 4) + \cos 4t[e^{-2t}(-2)] = -2e^{-2t}(2\sin 4t + \cos 4t)$

30.

$$f(t) = \sqrt{\frac{t}{t^2+4}} = \left(\frac{t}{t^2+4}\right)^{1/2} \quad \Rightarrow$$

$$f'(t) = \frac{1}{2}\left(\frac{t}{t^2+4}\right)^{-1/2} \cdot \frac{d}{dt}\left(\frac{t}{t^2+4}\right) = \frac{1}{2}\left(\frac{t^2+4}{t}\right)^{1/2} \cdot \frac{(t^2+4)(1)-t(2t)}{(t^2+4)^2}$$

$$= \frac{(t^2+4)^{1/2}}{2t^{1/2}} \cdot \frac{t^2+4-2t^2}{(t^2+4)^2} = \frac{4-t^2}{2t^{1/2}(t^2+4)^{3/2}}$$

34.

$$y = \cos\sqrt{\sin(\tan \pi x)} = \cos(\sin(\tan \pi x))^{1/2} \quad \Rightarrow$$

$$y' = -\sin(\sin(\tan \pi x))^{1/2} \cdot \frac{d}{dx}(\sin(\tan \pi x))^{1/2} = -\sin(\sin(\tan \pi x))^{1/2} \cdot \tfrac{1}{2}(\sin(\tan \pi x))^{-1/2} \cdot \frac{d}{dx}(\sin(\tan \pi x))$$

$$= \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \frac{d}{dx}\tan \pi x = \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \sec^2(\pi x) \cdot \pi$$

$$= \frac{-\pi \cos(\tan \pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$$

56.

(a) $h(x) = f(f(x)) \quad \Rightarrow \quad h'(x) = f'(f(x))f'(x)$. So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \quad \Rightarrow \quad g'(x) = f'(x^2) \cdot \frac{d}{dx}(x^2) = f'(x^2)(2x)$. So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(2) = 8$.

64.

$$F(x) = f(xf(xf(x))) \quad \Rightarrow$$

$$F'(x) = f'(xf(xf(x))) \cdot \frac{d}{dx}(xf(xf(x))) = f'(xf(xf(x))) \cdot \left[x \cdot f'(xf(x)) \cdot \frac{d}{dx}(xf(x)) + f(xf(x)) \cdot 1\right]$$

$$= f'(xf(xf(x))) \cdot [xf'(xf(x)) \cdot (xf'(x) + f(x) \cdot 1) + f(xf(x))], \quad \text{so}$$

$$F'(1) = f'(f(f(1))) \cdot [f'(f(1)) \cdot (f'(1) + f(1)) + f(f(1))] = f'(f(2)) \cdot [f'(2) \cdot (4+2) + f(2)]$$

$$= f'(3) \cdot [5 \cdot 6 + 3] = 6 \cdot 33 = 198.$$

72.

$L(t) = 12 + 2.8\sin\left(\frac{2\pi}{365}(t-80)\right) \quad \Rightarrow \quad L'(t) = 2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\left(\frac{2\pi}{365}\right)$.

On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

76.

(a) The derivative dV/dr represents the rate of change of the volume with respect to the radius and the derivative dV/dt represents the rate of change of the volume with respect to time.

(b) Since $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$.