Homework 5 - Solutions

Homework scores are out of 30 points.

Please check that your solutions are correct on the ungraded problems.

Section 3.5

2. (a) $\frac{d}{dx}\left(\cos x + \sqrt{y}\right) = \frac{d}{dx}(5) \Rightarrow -\sin x + \frac{1}{2}y^{-1/2} \cdot y' = 0 \Rightarrow \frac{1}{2\sqrt{y}} \cdot y' = \sin x \Rightarrow y' = 2\sqrt{y}\sin x$

(b) $\cos x + \sqrt{y} = 5 \Rightarrow \sqrt{y} = 5 - \cos x \Rightarrow y = (5 - \cos x)^2$, so $y' = 2(5 - \cos x)'(\sin x) = 2\sin x(5 - \cos x)$.

(c) From part (a), $y' = 2\sqrt{y}\sin x = 2\sqrt{(5-\cos x)^2} = 2(5 - \cos x)\sin x$ [since $5 - \cos x > 0$].

14. $\tan(x - y) = \frac{y}{1 + x^2} \Rightarrow (1 + x^2)\tan(x - y) = y \Rightarrow (1 + x^2)\sec^2(x - y) \cdot (1 - y') + \tan(x - y) \cdot 2x = y' \Rightarrow$

$(1 + x^2)\sec^2(x - y) - (1 + x^2)\sec^2(x - y) \cdot y' + 2x\tan(x - y) = y' \Rightarrow$

$(1 + x^2)\sec^2(x - y) + 2x\tan(x - y) = \left[1 + (1 + x^2)\sec^2(x - y)\right] \cdot y' \Rightarrow$

$$y' = \frac{(1 + x^2)\sec^2(x - y) + 2x\tan(x - y)}{1 + (1 + x^2)\sec^2(x - y)}$$

18. $\frac{d}{dx}[g(x) + x\sin g(x)] = \frac{d}{dx}(x^2) \Rightarrow g'(x) + x\cos g(x) \cdot g'(x) + \sin g(x) \cdot 1 = 2x$. If $x = 0$, we have

$g'(0) + 0 + \sin g(0) = 2(0) \Rightarrow g'(0) + \sin 0 = 0 \Rightarrow g'(0) + 0 = 0 \Rightarrow g'(0) = 0$.

28. $y^2(y^2 - 4) = x^2(x^2 - 5) \Rightarrow y^4 - 4y^2 = x^4 - 5x^2 \Rightarrow 4y^3 y' - 8y\,y' = 4x^3 - 10x$.

When $x = 0$ and $y = -2$, we have $-32y' + 16y' = 0 \Rightarrow -16y' = 0 \Rightarrow y' = 0$, so an equation of the tangent line is $y + 2 = 0(x - 0)$ or $y = -2$.

32. $\sqrt{x} + \sqrt{y} = 1 \Rightarrow \frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0 \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}} \Rightarrow$

$$y'' = -\frac{\sqrt{x}\left[\frac{1}{2\sqrt{y}}\right]y' - \sqrt{y}\left[\frac{1}{2\sqrt{x}}\right]}{x} = -\frac{\sqrt{x}\left(\frac{1}{\sqrt{y}}\right)\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) - \sqrt{y}\left(\frac{1}{\sqrt{x}}\right)}{2x} = \frac{1 + \frac{\sqrt{y}}{\sqrt{x}}}{2x}$$

$= \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}} = \frac{1}{2x\sqrt{x}}$ since x and y must satisfy the original equation, $\sqrt{x} + \sqrt{y} = 1$.

40. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\Rightarrow\quad \frac{2x}{a^2}+\frac{2yy'}{b^2}=0 \quad\Rightarrow\quad y'=-\frac{b^2x}{a^2y} \quad\Rightarrow$ an equation of the tangent line at (x_0,y_0) is

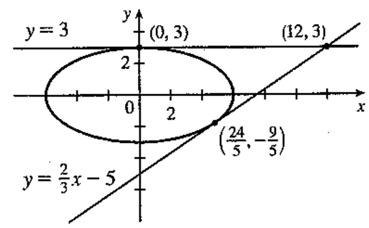
$y-y_0=\frac{-b^2x_0}{a^2y_0}(x-x_0)$. Multiplying both sides by $\frac{y_0}{b^2}$ gives $\frac{y_0y}{b^2}-\frac{y_0^2}{b^2}=-\frac{x_0x}{a^2}+\frac{x_0^2}{a^2}$. Since (x_0,y_0) lies on the ellipse,

we have $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$.

54. $x^2+4y^2=36 \quad\Rightarrow\quad 2x+8yy'=0 \quad\Rightarrow\quad y'=-\frac{x}{4y}$. Let (a,b) be a point on $x^2+4y^2=36$ whose tangent line passes through $(12,3)$. The tangent line is then $y-3=-\frac{a}{4b}(x-12)$, so $b-3=-\frac{a}{4b}(a-12)$. Multiplying both sides by $4b$ gives $4b^2-12b=-a^2+12a$, so $4b^2+a^2=12(a+b)$. But $4b^2+a^2=36$, so $36=12(a+b) \quad\Rightarrow\quad a+b=3 \quad\Rightarrow$ $b=3-a$. Substituting $3-a$ for b into $a^2+4b^2=36$ gives $a^2+4(3-a)^2=36 \quad\Leftrightarrow\quad a^2+36-24a+4a^2=36 \quad\Leftrightarrow$ $5a^2-24a=0 \quad\Leftrightarrow\quad a(5a-24)=0$, so $a=0$ or $a=\frac{24}{5}$. If $a=0$, $b=3-0=3$, and if $a=\frac{24}{5}$, $b=3-\frac{24}{5}=-\frac{9}{5}$.

So the two points on the ellipse are $(0,3)$ and $\left(\frac{24}{5},-\frac{9}{5}\right)$. Using $y-3=-\frac{a}{4b}(x-12)$ with $(a,b)=(0,3)$ gives us the tangent line $y-3=0$ or $y=3$. With $(a,b)=\left(\frac{24}{5},-\frac{9}{5}\right)$, we have $y-3=-\frac{24/5}{4(-9/5)}(x-12) \quad\Leftrightarrow\quad y-3=\frac{2}{3}(x-12) \quad\Leftrightarrow\quad y=\frac{2}{3}x-5$.

A graph of the ellipse and the tangent lines confirms our results.



Section 3.6

20. $F(\theta)=\arcsin\sqrt{\sin\theta}=\arcsin(\sin\theta)^{1/2} \quad\Rightarrow$

$$F'(\theta)=\frac{1}{\sqrt{1-\left(\sqrt{\sin\theta}\right)^2}}\cdot\frac{d}{d\theta}(\sin\theta)^{1/2}=\frac{1}{\sqrt{1-\sin\theta}}\cdot\frac{1}{2}(\sin\theta)^{-1/2}\cdot\cos\theta=\frac{\cos\theta}{2\sqrt{1-\sin\theta}\,\sqrt{\sin\theta}}$$

32. $\tan^{-1}(xy)=1+x^2y \quad\Rightarrow\quad \frac{1}{1+x^2y^2}(xy'+y\cdot 1)=0+x^2y'+2xy \quad\Rightarrow$

$$y'\left(\frac{x}{1+x^2y^2}-x^2\right)=2xy-\frac{y}{1+x^2y^2} \quad\Rightarrow$$

$$y'=\frac{2xy-\dfrac{y}{1+x^2y^2}}{\dfrac{x}{1+x^2y^2}-x^2}=\frac{2xy(1+x^2y^2)-y}{x-x^2(1+x^2y^2)}=\frac{y(-1-2x-2x^3y^2)}{x(1-x-x^3y^2)}.$$

38. Let $t=\frac{1+x^2}{1+2x^2}$. As $x\to\infty$, $t=\frac{1+x^2}{1+2x^2}=\frac{1/x^2+1}{1/x^2+2}\to\frac{1}{2}$.

$$\lim_{x\to\infty}\arccos\left(\frac{1+x^2}{1+2x^2}\right)=\lim_{t\to 1/2}\arccos t=\arccos\tfrac{1}{2}=\tfrac{\pi}{3}.$$

Section 3.7

18. $y = [\ln(1+e^x)]^2 \quad \Rightarrow \quad y' = 2[\ln(1+e^x)] \cdot \dfrac{1}{1+e^x} \cdot e^x = \dfrac{2e^x \ln(1+e^x)}{1+e^x}$

22. $y = \dfrac{\ln x}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3} \quad \Rightarrow$

$y'' = \dfrac{x^3(-2/x) - (1-2\ln x)(3x^2)}{(x^3)^2} = \dfrac{x^2(-2-3+6\ln x)}{x^6} = \dfrac{6\ln x - 5}{x^4}$

36. $y = \sqrt[4]{\dfrac{x^2+1}{x^2-1}} \quad \Rightarrow \quad \ln y = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1) \quad \Rightarrow \quad \dfrac{1}{y}y' = \dfrac{1}{4} \cdot \dfrac{1}{x^2+1} \cdot 2x - \dfrac{1}{4} \cdot \dfrac{1}{x^2-1} \cdot 2x \quad \Rightarrow$

$y' = \sqrt[4]{\dfrac{x^2+1}{x^2-1}} \cdot \dfrac{1}{2}\left(\dfrac{x}{x^2+1} - \dfrac{x}{x^2-1}\right) = \dfrac{1}{2}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}\left(\dfrac{-2x}{x^4-1}\right) = \dfrac{x}{1-x^4}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}$

42. $y = (\sin x)^{\ln x} \quad \Rightarrow \quad \ln y = \ln(\sin x)^{\ln x} \quad \Rightarrow \quad \ln y = \ln x \cdot \ln \sin x \quad \Rightarrow \quad \dfrac{1}{y}y' = \ln x \cdot \dfrac{1}{\sin x} \cdot \cos x + \ln \sin x \cdot \dfrac{1}{x} \quad \Rightarrow$

$y' = y\left(\ln x \cdot \dfrac{\cos x}{\sin x} + \dfrac{\ln \sin x}{x}\right) \quad \Rightarrow \quad y' = (\sin x)^{\ln x}\left(\ln x \cot x + \dfrac{\ln \sin x}{x}\right)$

44. $x^y = y^x \quad \Rightarrow \quad y \ln x = x \ln y \quad \Rightarrow \quad y \cdot \dfrac{1}{x} + (\ln x) \cdot y' = x \cdot \dfrac{1}{y} \cdot y' + \ln y \quad \Rightarrow \quad y' \ln x - \dfrac{x}{y}y' = \ln y - \dfrac{y}{x} \quad \Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

Section 3.9

4. Let $A = \dfrac{N(1985) - N(1990)}{1985 - 1990} = \dfrac{17.04 - 19.33}{-5} = 0.458$ and $B = \dfrac{N(1995) - N(1990)}{1995 - 1990} =$

$\dfrac{21.91 - 19.33}{5} = 0.516$. Then $N'(1990) = \lim\limits_{t \to 1990} \dfrac{N(t) - N(1990)}{t - 1990} \approx \dfrac{A + B}{2} = 0.487$ million/year.

So $N(1989) \approx N(1990) + N'(1990)(1989 - 1990) \approx 19.33 + 0.487(-1) = 18.843$ million.

$N'(2005) \approx \dfrac{N(2000) - N(2005)}{2000 - 2005} = \dfrac{24.70 - 27.68}{-5} = 0.596$ million/year.

$N(2010) \approx N(2005) + N'(2005)(2010 - 2005) \approx 27.68 + 0.596(5) = 30.66$ million.

10. $g(x) = \sqrt[3]{1 + x} = (1 + x)^{1/3} \Rightarrow g'(x) = \frac{1}{3}(1 + x)^{-2/3}$, so $g(0) = 1$ and $g'(0) = \frac{1}{3}$. Therefore, $\sqrt[3]{1 + x} = g(x) \approx g(0) + g'(0)(x - 0) = 1 + \frac{1}{3}x$.

So $\sqrt[3]{0.95} = \sqrt[3]{1 + (-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\overline{3}$,

and $\sqrt[3]{1.1} = \sqrt[3]{1 + 0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\overline{3}$.

16. To estimate $e^{-0.015}$, we'll find the linearization of $f(x) = e^x$ at $a = 0$. Since $f'(x) = e^x$, $f(0) = 1$, and $f'(0) = 1$, we have $L(x) = 1 + 1(x - 0) = x + 1$. Thus, $e^x \approx x + 1$ when x is near 0, so $e^{-0.015} \approx -0.015 + 1 = 0.985$.

28. (a) $A = \pi r^2 \Rightarrow dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30\text{ cm}^2$.

(b) Relative error $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\,dr}{\pi r^2} = \dfrac{2\,dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error $=$ relative error $\times 100\% = 0.01\overline{6} \times 100\% = 1.\overline{6}\%$.

36. (a) $g'(x) = \sqrt{x^2 + 5} \Rightarrow g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95 - 2) = -4 + 3(-0.05) = -4.15$.

$g(2.05) \approx g(2) + g'(2)(2.05 - 2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2 + 5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.