Math 19: Calculus Summer 2009

Homework 6 - Solutions

Homework scores are out of 30 points.

Please check that your solutions are correct on the ungraded problems.

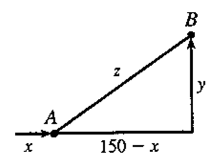
Section 4.1

2. (a) $A = \pi r^2 \;\Rightarrow\; \dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = 2\pi r\,\dfrac{dr}{dt}$ (b) $\dfrac{dA}{dt} = 2\pi r\,\dfrac{dr}{dt} = 2\pi(30\text{ m})(1\text{ m/s}) = 60\pi\text{ m}^2/\text{s}$

12. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

(b) Unknown: the rate at which the distance between the ships is changing at 4:00 PM. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

(c)



(d) $z^2 = (150 - x)^2 + y^2 \;\Rightarrow\; 2z\,\dfrac{dz}{dt} = 2(150 - x)\left(-\dfrac{dx}{dt}\right) + 2y\,\dfrac{dy}{dt}$

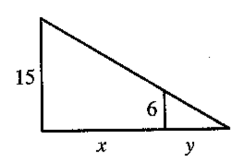
(e) At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \;\Rightarrow\; z = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10{,}100}$.

So $\dfrac{dz}{dt} = \dfrac{1}{z}\left[(x - 150)\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right] = \dfrac{-10(35) + 100(25)}{\sqrt{10{,}100}} = \dfrac{215}{\sqrt{101}} \approx 21.4$ km/h.

14. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5$ ft/s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\dfrac{d}{dt}(x + y)$ when $x = 40$ ft.

(c)

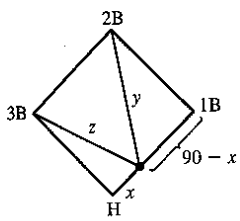


(d) By similar triangles, $\dfrac{15}{6} = \dfrac{x + y}{y} \;\Rightarrow\; 15y = 6x + 6y \;\Rightarrow\; 9y = 6x \;\Rightarrow\; y = \tfrac{2}{3}x$.

(e) The tip of the shadow moves at a rate of $\dfrac{d}{dt}(x + y) = \dfrac{d}{dt}\left(x + \dfrac{2}{3}x\right) = \dfrac{5}{3}\dfrac{dx}{dt} = \tfrac{5}{3}(5) = \tfrac{25}{3}$ ft/s.

18. We are given that $\frac{dx}{dt} = 24$ ft/s.

(a)



$y^2 = (90 - x)^2 + 90^2 \Rightarrow 2y\frac{dy}{dt} = 2(90 - x)\left(-\frac{dx}{dt}\right)$. When $x = 45$,

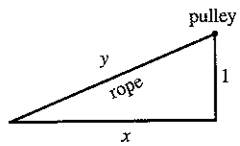
$y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so $\frac{dy}{dt} = \frac{90 - x}{y}\left(-\frac{dx}{dt}\right) = \frac{45}{45\sqrt{5}}(-24) = -\frac{24}{\sqrt{5}}$,

so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ ft/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer—and we do.

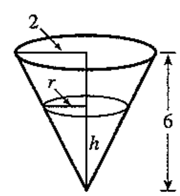
$z^2 = x^2 + 90^2 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt}$. When $x = 45$, $z = 45\sqrt{5}$, so $\frac{dz}{dt} = \frac{45}{45\sqrt{5}}(24) = \frac{24}{\sqrt{5}} \approx 10.7$ ft/s.

20.



Given $\frac{dy}{dt} = -1$ m/s, find $\frac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \Rightarrow 2y\frac{dy}{dt} = 2x\frac{dx}{dt} \Rightarrow$ $\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt} = -\frac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so $\frac{dx}{dt} = -\frac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\frac{\sqrt{65}}{8} \approx 1.01$ m/s.

26. If $C =$ the rate at which water is pumped in, then $\frac{dV}{dt} = C - 10{,}000$, where

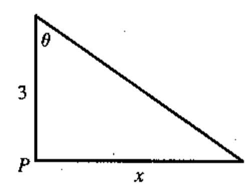


$V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles, $\frac{r}{2} = \frac{h}{6} \Rightarrow r = \frac{1}{3}h \Rightarrow$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$. When $h = 200$ cm,

$\frac{dh}{dt} = 20$ cm/min, so $C - 10{,}000 = \frac{\pi}{9}(200)^2(20) \Rightarrow C = 10{,}000 + \frac{800{,}000}{9}\pi \approx 289{,}253$ cm^3/min.

38. We are given that $\frac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3\tan\theta \Rightarrow$



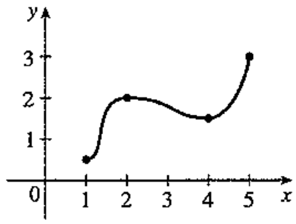
$\frac{dx}{dt} = 3\sec^2\theta\,\frac{d\theta}{dt}$. When $x = 1$, $\tan\theta = \frac{1}{3}$, so $\sec^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$

and $\frac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80}{3}\pi \approx 83.8$ km/min.

Section 4.2

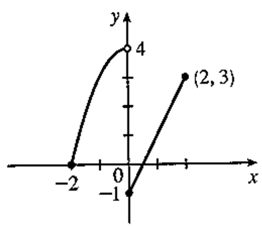
6. There is no absolute maximum value; absolute minimum value is $g(4) = 1$; local maximum values are $g(3) = 4$ and $g(6) = 3$; local minimum values are $g(2) = 2$ and $g(4) = 1$.

8. Absolute minimum at 1, absolute maximum at 5,
local maximum at 2, local minimum at 4



22. $f(x) = \begin{cases} 4 - x^2 & \text{if } -2 \le x < 0 \\ 2x - 1 & \text{if } 0 \le x \le 2 \end{cases}$

Abolute minimum $f(0) = -1$; no local minimim.
No absolute or local maximum.



38. $f(x) = x^{-2} \ln x \;\Rightarrow\; f'(x) = x^{-2}(1/x) + (\ln x)(-2x^{-3}) = x^{-3} - 2x^{-3}\ln x = x^{-3}(1 - 2\ln x) = \dfrac{1 - 2\ln x}{x^3}$.

$f'(x) = 0 \;\Rightarrow\; 1 - 2\ln x = 0 \;\Rightarrow\; \ln x = \frac{1}{2} \;\Rightarrow\; x = e^{1/2} \approx 1.65$. $f'(0)$ does not exist, but 0 is not in the domain of f, so the only critical number is $\sqrt{e}$.

44. $f(x) = x^3 - 6x^2 + 9x + 2$, $[-1, 4]$. $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0 \;\Leftrightarrow\; x = 1, 3$. $f(-1) = -14$, $f(1) = 6$, $f(3) = 2$, and $f(4) = 6$. So $f(1) = f(4) = 6$ is the absolute maximum value and $f(-1) = -14$ is the absolute minimum value.

58

(b) $f(x) = e^{x^3 - x} \;\Rightarrow\; f'(x) = e^{x^3 - x}(3x^2 - 1)$. So $f'(x) = 0$ on $[-1, 0] \;\Rightarrow\; x = -\sqrt{1/3}$.

$f(-1) = f(0) = 1$ (minima) and $f\left(-\sqrt{1/3}\right) = e^{-\sqrt{3}/9 + \sqrt{3}/3} = e^{2\sqrt{3}/9}$ (maximum).

Section 4.3

8. (a) $f(x) = 4x^3 + 3x^2 - 6x + 1 \quad \Rightarrow \quad f'(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x - 1)(x + 1)$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $x > \frac{1}{2}$ and $f'(x) < 0 \quad \Leftrightarrow \quad -1 < x < \frac{1}{2}$. So f is increasing on $(-\infty, -1)$ and $\left(\frac{1}{2}, \infty\right)$ and f is decreasing on $\left(-1, \frac{1}{2}\right)$.

(b) f changes from increasing to decreasing at $x = -1$ and from decreasing to increasing at $x = \frac{1}{2}$. Thus, $f(-1) = 6$ is a local maximum value and $f\left(\frac{1}{2}\right) = -\frac{3}{4}$ is a local minimum value.

(c) $f''(x) = 24x + 6 = 6(4x + 1)$. $f''(x) > 0 \quad \Leftrightarrow \quad x > -\frac{1}{4}$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -\frac{1}{4}$. Thus, f is concave upward on $\left(-\frac{1}{4}, \infty\right)$ and concave downward on $\left(-\infty, -\frac{1}{4}\right)$. There is an inflection point at $\left(-\frac{1}{4}, f\left(-\frac{1}{4}\right)\right) = \left(-\frac{1}{4}, \frac{21}{8}\right)$.

14. (a) $f(x) = x^2 \ln x \quad \Rightarrow \quad f'(x) = x^2(1/x) + (\ln x)(2x) = x + 2x \ln x = x(1 + 2\ln x)$. The domain of f is $(0, \infty)$, so the sign of f' is determined solely by the factor $1 + 2\ln x$. $f'(x) > 0 \quad \Leftrightarrow \quad \ln x > -\frac{1}{2} \quad \Leftrightarrow \quad x > e^{-1/2}$ $[\approx 0.61]$ and $f'(x) < 0 \quad \Leftrightarrow \quad 0 < x < e^{-1/2}$. So f is increasing on $(e^{-1/2}, \infty)$ and f is decreasing on $(0, e^{-1/2})$.

(b) f changes from decreasing to increasing at $x = e^{-1/2}$. Thus, $f(e^{-1/2}) = (e^{-1/2})^2 \ln(e^{-1/2}) = e^{-1}(-1/2) = -1/(2e)$ $[\approx -0.18]$ is a local minimum value.

(c) $f'(x) = x(1 + 2\ln x) \quad \Rightarrow \quad f''(x) = x(2/x) + (1 + 2\ln x) \cdot 1 = 2 + 1 + 2\ln x = 3 + 2\ln x$. $f''(x) > 0 \quad \Leftrightarrow \quad 3 + 2\ln x > 0 \quad \Leftrightarrow \quad \ln x > -3/2 \quad \Leftrightarrow \quad x > e^{-3/2}$ $[\approx 0.22]$. Thus, f is concave upward on $(e^{-3/2}, \infty)$ and f is concave downward on $(0, e^{-3/2})$. $f(e^{-3/2}) = (e^{-3/2})^2 \ln e^{-3/2} = e^{-3}(-3/2) = -3/(2e^3)$ $[\approx -0.07]$. There is a point of inflection at $\left(e^{-3/2}, f(e^{-3/2})\right) = \left(e^{-3/2}, -3/(2e^3)\right)$.

18. $f(x) = \dfrac{x}{x^2 + 4} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2 + 4) \cdot 1 - x(2x)}{(x^2 + 4)^2} = \dfrac{4 - x^2}{(x^2 + 4)^2} = \dfrac{(2 + x)(2 - x)}{(x^2 + 4)^2}$.

First Derivative Test: $f'(x) > 0 \quad \Rightarrow \quad -2 < x < 2$ and $f'(x) < 0 \quad \Rightarrow \quad x > 2$ or $x < -2$. Since f' changes from positive to negative at $x = 2$, $f(2) = \frac{1}{4}$ is a local maximum value; and since f' changes from negative to positive at $x = -2$, $f(-2) = -\frac{1}{4}$ is a local minimum value.

20. (a) $f(x) = x^4(x - 1)^3 \quad \Rightarrow \quad f'(x) = x^4 \cdot 3(x - 1)^2 + (x - 1)^3 \cdot 4x^3 = x^3(x - 1)^2\,[3x + 4(x - 1)] = x^3(x - 1)^2(7x - 4)$
The critical numbers are 0, 1, and $\frac{4}{7}$.

(b)
$$\begin{aligned} f''(x) &= 3x^2(x - 1)^2(7x - 4) + x^3 \cdot 2(x - 1)(7x - 4) + x^3(x - 1)^2 \cdot 7 \\ &= x^2(x - 1)\,[3(x - 1)(7x - 4) + 2x(7x - 4) + 7x(x - 1)] \end{aligned}$$

Now $f''(0) = f''(1) = 0$, so the Second Derivative Test gives no information for $x = 0$ or $x = 1$.

$f''\left(\frac{4}{7}\right) = \left(\frac{4}{7}\right)^2\left(\frac{4}{7} - 1\right)\left[0 + 0 + 7\left(\frac{4}{7}\right)\left(\frac{4}{7} - 1\right)\right] = \left(\frac{4}{7}\right)^2\left(-\frac{3}{7}\right)(4)\left(-\frac{3}{7}\right) > 0$, so there is a local minimum at $x = \frac{4}{7}$.

(c) f' is positive on $(-\infty, 0)$, negative on $\left(0, \frac{4}{7}\right)$, positive on $\left(\frac{4}{7}, 1\right)$, and positive on $(1, \infty)$. So f has a local maximum at $x = 0$, a local minimum at $x = \frac{4}{7}$, and no local maximum or minimum at $x = 1$.

72. $f(x) = cx + \dfrac{1}{x^2+3} \quad \Rightarrow \quad f'(x) = c - \dfrac{2x}{(x^2+3)^2}$. $f'(x) > 0 \quad \Leftrightarrow \quad c > \dfrac{2x}{(x^2+3)^2}$ [call this $g(x)$].

Now f' is positive (and hence f increasing) if $c > g$, so we'll find the maximum value of g.

$$g'(x) = \frac{(x^2+3)^2 \cdot 2 - 2x \cdot 2(x^2+3) \cdot 2x}{[(x^2+3)^2]^2} = \frac{2(x^2+3)[(x^2+3) - 4x^2]}{(x^2+3)^4} = \frac{2(3-3x^2)}{(x^2+3)^3} = \frac{6(1+x)(1-x)}{(x^2+3)^3}.$$

$g'(x) = 0 \quad \Leftrightarrow \quad x = \pm 1$. $g'(x) > 0$ on $(0,1)$ and $g'(x) < 0$ on $(1,\infty)$, so g is increasing on $(0,1)$ and decreasing on $(1,\infty)$, and hence g has a maximum value on $(0,\infty)$ of $g(1) = \frac{2}{16} = \frac{1}{8}$. Also since $g(x) \le 0$ if $x \le 0$, the maximum value of g on $(-\infty,\infty)$ is $\frac{1}{8}$. Thus, when $c > \frac{1}{8}$, f is increasing. When $c = \frac{1}{8}$, $f'(x) > 0$ on $(-\infty, 1)$ and $(1,\infty)$, and hence f is increasing on these intervals. Since f is continuous, we may conclude that f is also increasing on $(-\infty,\infty)$ if $c = \frac{1}{8}$. Therefore, f is increasing on $(-\infty,\infty)$ if $c \ge \frac{1}{8}$.