Homework 7 - Solutions

Homework scores are out of 30 points.

Please check that your solutions are correct on the ungraded problems.

Section 4.3

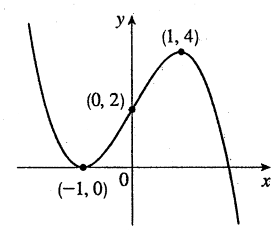
22. (a) $f(x) = 2 + 3x - x^3 \;\Rightarrow\; f'(x) = 3 - 3x^2 = -3(x^2 - 1) = -3(x+1)(x-1)$.

$f'(x) > 0 \;\Leftrightarrow\; -1 < x < 1$ and $f'(x) < 0 \;\Leftrightarrow\; x < -1$ or $x > 1$. So f is increasing on $(-1, 1)$ and f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) $f(-1) = 0$ is a local minimum value and $f(1) = 4$ is a local maximum value.

(c) $f''(x) = -6x \;\Rightarrow\; f''(x) > 0$ on $(-\infty, 0)$ and $f''(x) < 0$ on $(0, \infty)$. So f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$. There is an inflection point at $(0, 2)$.

(d)



34. $f(x) = \dfrac{x^2}{(x-2)^2}$ has domain $(-\infty, 2) \cup (2, \infty)$.

(a) $$\lim_{x\to\pm\infty} \frac{x^2}{x^2 - 4x + 4} = \lim_{x\to\pm\infty} \frac{x^2/x^2}{(x^2 - 4x + 4)/x^2} = \lim_{x\to\pm\infty} \frac{1}{1 - 4/x + 4/x^2} = \frac{1}{1 - 0 + 0} = 1,$$

so $y = 1$ is a HA. $\displaystyle\lim_{x\to 2^+} \frac{x^2}{(x-2)^2} = \infty$ since $x^2 \to 4$ and $(x-2)^2 \to 0^+$ as $x \to 2^+$, so $x = 2$ is a VA.

(b) $$f(x) = \frac{x^2}{(x-2)^2} \;\Rightarrow\; f'(x) = \frac{(x-2)^2(2x) - x^2 \cdot 2(x-2)}{[(x-2)^2]^2} = \frac{2x(x-2)[(x-2) - x]}{(x-2)^4} = \frac{-4x}{(x-2)^3}.$$

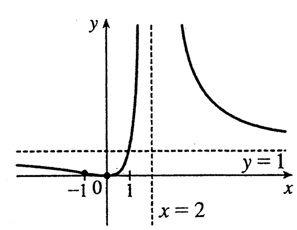
$f'(x) > 0$ if $0 < x < 2$ and $f'(x) < 0$ if $x < 0$ or $x > 2$, so f is increasing on $(0, 2)$ and f is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

(c) $f(0) = 0$ is a local minimum value.

(d) $$f''(x) = \frac{(x-2)^3(-4) - (-4x)\cdot 3(x-2)^2}{[(x-2)^3]^2} = \frac{4(x-2)^2[-(x-2) + 3x]}{(x-2)^6} = \frac{8(x+1)}{(x-2)^4}$$

$f''(x) > 0$ if $x > -1$ $(x \neq 2)$ and $f''(x) < 0$ if $x < -1$. Thus, f is CU on $(-1, 2)$ and $(2, \infty)$, and f is CD on $(-\infty, -1)$. There is an inflection point at $\left(-1, \frac{1}{9}\right)$.

(e)



38. $f(x) = \dfrac{e^x}{1+e^x}$ has domain $\mathbb{R}$.

(a) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{e^x/e^x}{(1+e^x)/e^x} = \lim\limits_{x\to\infty} \dfrac{1}{e^{-x}+1} = \dfrac{1}{0+1} = 1$, so $y = 1$ is a HA.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{e^x}{1+e^x} = \dfrac{0}{1+0} = 0$, so $y = 0$ is a HA. No VA.

(b) $f'(x) = \dfrac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} = \dfrac{e^x}{(1+e^x)^2} > 0$ for all x. Thus, f is increasing on $\mathbb{R}$.

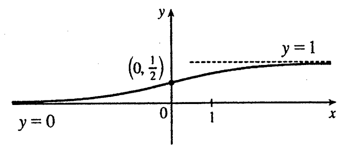
(c) There is no local maximum or minimum.

(d) $f''(x) = \dfrac{(1+e^x)^2 e^x - e^x \cdot 2(1+e^x)e^x}{[(1+e^x)^2]^2}$

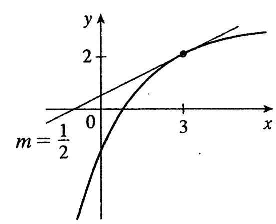
$= \dfrac{e^x(1+e^x)[(1+e^x)-2e^x]}{(1+e^x)^4} = \dfrac{e^x(1-e^x)}{(1+e^x)^3}$

$f''(x) > 0 \iff 1 - e^x > 0 \iff x < 0$, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. There is an inflection point at $\left(0, \frac{1}{2}\right)$.

(e)



50. (a) $f(3) = 2 \Rightarrow$ the point $(3, 2)$ is on the graph of f. $f'(3) = \frac{1}{2} \Rightarrow$ the slope of the tangent line at $(3, 2)$ is $\frac{1}{2}$. $f'(x) > 0$ for all $x \Rightarrow f$ is increasing on $\mathbb{R}$. $f''(x) < 0$ for all $x \Rightarrow f$ is concave downward on $\mathbb{R}$. A possible graph for f is shown.



(b) The tangent line at $(3, 2)$ has equation $y - 2 = \frac{1}{2}(x-3)$, or $y = \frac{1}{2}x + \frac{1}{2}$, and x-intercept -1. Since f is concave downward on $\mathbb{R}$, f is below the x-axis at $x = -1$, and hence changes sign at least once. Since f is increasing on $\mathbb{R}$, it changes sign at most once. Thus, it changes sign exactly once and there is one solution of the equation $f(x) = 0$.

(c) $f'' < 0 \Rightarrow f'$ is decreasing. Since $f'(3) = \frac{1}{2}$, $f'(2)$ must be greater than $\frac{1}{2}$, so no, it is not possible that $f'(2) = \frac{1}{3}$.

60. $f(x) = axe^{bx^2} \Rightarrow f'(x) = a\left[xe^{bx^2} \cdot 2bx + e^{bx^2} \cdot 1\right] = ae^{bx^2}(2bx^2+1)$. For $f(2) = 1$ to be a maximum value, we must have $f'(2) = 0$. $f(2) = 1 \Rightarrow 1 = 2ae^{4b}$ and $f'(2) = 0 \Rightarrow 0 = (8b+1)ae^{4b}$. So $8b + 1 = 0$ $[a \neq 0] \Rightarrow b = -\frac{1}{8}$ and now $1 = 2ae^{-1/2} \Rightarrow a = \sqrt{e}/2$.

Section 4.5

Note: In the following solutions, the use of L'Hôpital's rule is indicated by an "H" above the equal sign.

12. $\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{\csc\theta} = \frac{0}{1} = 0$. L'Hospital's Rule does not apply.

24. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{1+1}{1} = 2$

28. This limit has the form $\infty \cdot 0$. $\lim_{x\to -\infty} x^2 e^x = \lim_{x\to -\infty} \frac{x^2}{e^{-x}} \overset{\text{H}}{=} \lim_{x\to -\infty} \frac{2x}{-e^{-x}} \overset{\text{H}}{=} \lim_{x\to -\infty} \frac{2}{e^{-x}} = \lim_{x\to -\infty} 2e^x = 0$

30. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x\to 0+} \sin x \ln x = \lim_{x\to 0+} \frac{\ln x}{\csc x} \overset{\text{H}}{=} \lim_{x\to 0+} \frac{1/x}{-\csc x \cot x} = -\lim_{x\to 0+} \left(\frac{\sin x}{x} \cdot \tan x\right) = -\left(\lim_{x\to 0+} \frac{\sin x}{x}\right)\left(\lim_{x\to 0+} \tan x\right)$$

$$= -1 \cdot 0 = 0$$

38. As $x \to \infty$, $1/x \to 0$, and $e^{1/x} \to 1$. So the limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

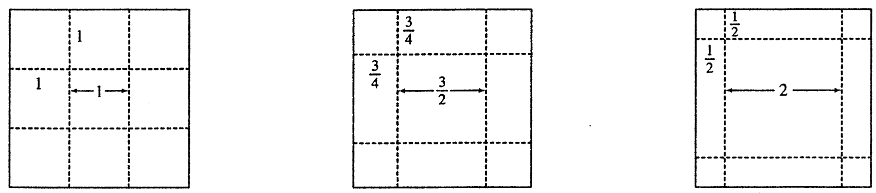
$$\lim_{x\to\infty} (xe^{1/x} - x) = \lim_{x\to\infty} x(e^{1/x} - 1) = \lim_{x\to\infty} \frac{e^{1/x} - 1}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} e^{1/x} = e^0 = 1$$

44. $y = x^{(\ln 2)/(1+\ln x)} \Rightarrow \ln y = \frac{\ln 2}{1+\ln x} \ln x \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{(\ln 2)(\ln x)}{1+\ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{(\ln 2)(1/x)}{1/x} = \lim_{x\to\infty} \ln 2 = \ln 2, \text{ so } \lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2.$$

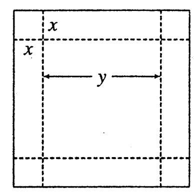
Section 4.6

10. (a)



The volumes of the resulting boxes are 1, 1.6875, and 2 ft^3. There appears to be a maximum volume of at least 2 ft^3.

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.



(c) Volume V = length × width × height $\Rightarrow$ $V = y \cdot y \cdot x = xy^2$

(d) Length of cardboard $= 3$ $\Rightarrow$ $x + y + x = 3$ $\Rightarrow$ $y + 2x = 3$

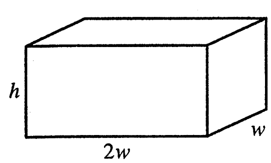
(e) $y + 2x = 3$ $\Rightarrow$ $y = 3 - 2x$ $\Rightarrow$ $V(x) = x(3 - 2x)^2$

(f) $V(x) = x(3 - 2x)^2$ $\Rightarrow$

$V'(x) = x \cdot 2(3 - 2x)(-2) + (3 - 2x)^2 \cdot 1 = (3 - 2x)[-4x + (3 - 2x)] = (3 - 2x)(-6x + 3)$,

so the critical numbers are $x = \frac{3}{2}$ and $x = \frac{1}{2}$. Now $0 \le x \le \frac{3}{2}$ and $V(0) = V\left(\frac{3}{2}\right) = 0$, so the maximum is $V\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(2)^2 = 2\ \text{ft}^3$, which is the value found from our third figure in part (a).

14.



$V = lwh$ $\Rightarrow$ $10 = (2w)(w)h = 2w^2h$, so $h = 5/w^2$.

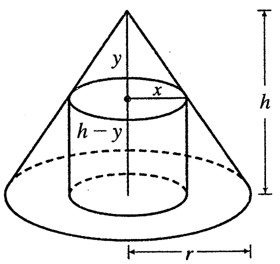
The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] = 20w^2 + 36wh$, so

$C(w) = 20w^2 + 36w(5/w^2) = 20w^2 + 180/w$.

$C'(w) = 40w - 180/w^2 = 40\left(w^3 - \frac{9}{2}\right)/w^2$ $\Rightarrow$ $w = \sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C when $w = \sqrt[3]{\frac{9}{2}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{9}{2}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{9}{2}}$.

$$C\left(\sqrt[3]{\tfrac{9}{2}}\right) = 20\left(\sqrt[3]{\tfrac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{9/2}} \approx \$163.54.$$

24.



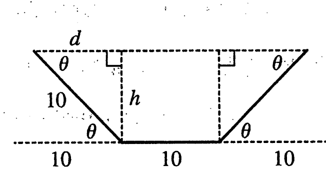
By similar triangles, $y/x = h/r$, so $y = hx/r$. The volume of the cylinder is $\pi x^2(h - y) = \pi hx^2 - (\pi h/r)x^3 = V(x)$. Now $V'(x) = 2\pi hx - (3\pi h/r)x^2 = \pi hx(2 - 3x/r)$. So $V'(x) = 0 \Rightarrow x = 0$ or $x = \frac{2}{3}r$. The maximum clearly occurs when $x = \frac{2}{3}r$ and then the volume is

$$\pi hx^2 - (\pi h/r)x^3 = \pi hx^2(1 - x/r) = \pi\left(\tfrac{2}{3}r\right)^2 h\left(1 - \tfrac{2}{3}\right) = \tfrac{4}{27}\pi r^2 h.$$

46. (a) Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y = p(x)$ passes through $(20, 10)$ and $(18, 11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y - 10 = -\frac{1}{2}(x - 20) \Leftrightarrow y = -\frac{1}{2}x + 20$. Thus, the demand is $p(x) = -\frac{1}{2}x + 20$ and the revenue is $R(x) = xp(x) = -\frac{1}{2}x^2 + 20x$.

(b) The cost is $C(x) = 6x$, so the profit is $P(x) = R(x) - C(x) = -\frac{1}{2}x^2 + 14x$. Then $0 = P'(x) = -x + 14 \Rightarrow x = 14$. Since $P''(x) = -1 < 0$, the selling price for maximum profit is $p(14) = -\frac{1}{2}(14) + 20 = \$13$.

56. We maximize the cross-sectional area



$$A(\theta) = 10h + 2\left(\tfrac{1}{2}dh\right) = 10h + dh = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta)$$
$$= 100(\sin\theta + \sin\theta\,\cos\theta),\ 0 \le \theta \le \tfrac{\pi}{2}$$
$$A'(\theta) = 100(\cos\theta + \cos^2\theta - \sin^2\theta) = 100(\cos\theta + 2\cos^2\theta - 1)$$
$$= 100(2\cos\theta - 1)(\cos\theta + 1) = 0 \text{ when } \cos\theta = \tfrac{1}{2} \Leftrightarrow \theta = \tfrac{\pi}{3} \quad [\cos\theta \ne -1 \text{ since } 0 \le \theta \le \tfrac{\pi}{2}.]$$

Now $A(0) = 0$, $A\left(\frac{\pi}{2}\right) = 100$ and $A\left(\frac{\pi}{3}\right) = 75\sqrt{3} \approx 129.9$, so the maximum occurs when $\theta = \frac{\pi}{3}$.