Quiz 2 Solutions

1. *(2 points)* Find the limit
$$\lim_{x\to\infty} \frac{x+1}{2x-1}\,.$$

Solution. This is a limit at infinity. So we should divide both the numerator and the denominator by the highest power of x in the denominator.

$$\lim_{x\to\infty} \frac{x+1}{2x-1} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{2-\frac{1}{x}} = \frac{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \frac{1}{x}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \frac{1}{x}} = \frac{1+0}{2-0} = \frac{1}{2}\,.$$

2. *(2 points)* Find the limit
$$\lim_{x\to 1} \frac{x^2-2x+1}{x-1}\,.$$

Solution. Direct substitution gives $\frac{0}{0}$ which doesn't give any information about the limit (this form is called an indeterminate form). But we can simplify the fraction before evaluating the limit as follows.

$$\lim_{x\to 1} \frac{x^2-2x+1}{x-1} = \lim_{x\to 1} \frac{(x-1)^2}{x-1} = \lim_{x\to 1}(x-1) = 1-1 = 0\,.$$

3. *(2 points)* Find the limit
$$\lim_{x\to 0} x^2 \sin^2\left(\ln(x^2)\right).$$

Show your reasoning briefly. You may use the fact that

$$0 \leqslant x^2 \sin^2\left(\ln(x^2)\right) \leqslant x^2$$

for all $x \neq 0$ (you don't need to verify this fact).

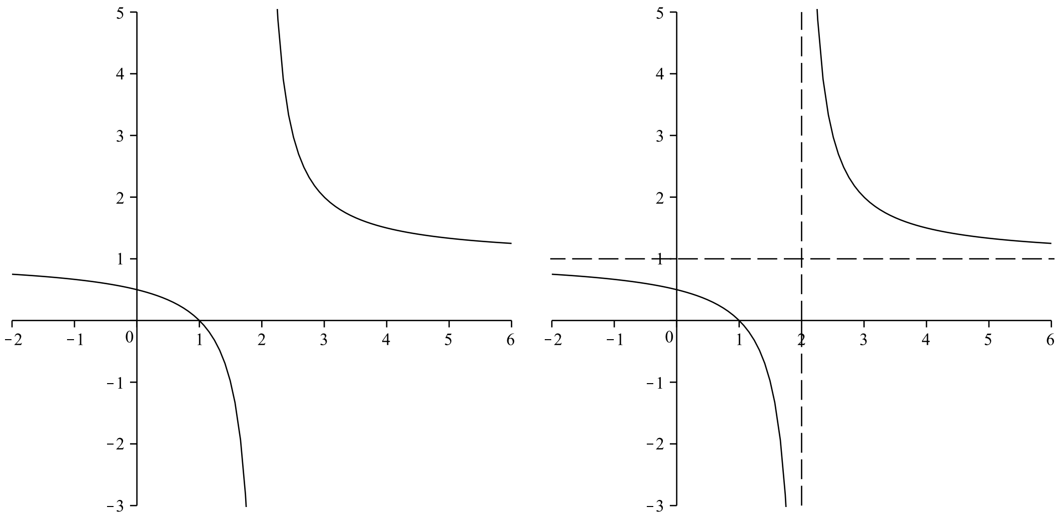
Solution. It's easy to observe that $\lim\limits_{x\to 0} 0 = 0$ and $\lim\limits_{x\to 0} x^2 = 0$. So by Squeeze Theorem, the limit of any function between these two functions is 0. In particular, we have

$$\lim_{x\to 0} x^2 \sin^2\left(\ln(x^2)\right) = 0\,.$$

4. *(2 points)* Let

$$f(x) = \frac{x-1}{x-2}.$$

The graph of $f(x)$ is shown as below. Write down the equations for all horizontal and vertical asymptotes of $f(x)$, and sketch them on the graph. You don't need to justify your answer.



Solution. The only horizontal asymptote is $y = 1$ and the only vertical asymptote is $x = 2$. The reason is as follows.

To find the horizontal asymptote, we need to evaluate $\lim_{x\to\infty} \frac{x-1}{x-2}$ and $\lim_{x\to-\infty} \frac{x-1}{x-2}$. Since both of the limits are 1, we get the only horizontal asymptote $y = 1$.

To find the vertical asymptote, we need to find all infinite discontinuities of the function. It's easy to see that this function is continuous everywhere except $x = 2$. So the only candidate is $x = 2$. Simple calculation shows that $\lim_{x\to 2-} \frac{x-1}{x-2} = -\infty$ and $\lim_{x\to 2+} \frac{x-1}{x-2} = \infty$ (do the calculations yourself!). Since the limits involve infinity, $x = 2$ is a vertical asymptote.

The sketch of the asymptotes are shown as above.

5. *(2 points)* Let

$$f(x) = \begin{cases} 2x & \text{if } x < 1; \\ 1 & \text{if } x = 1; \\ \ln x & \text{if } x > 1. \end{cases}$$

Determine whether $f(x)$ is continuous at $x = 1$. Show your reasoning briefly.

Solution. This function is not continuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist. In fact, we have

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} 2x = 2$$

and

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} \ln x = 0\,.$$

Since the left-hand and right-hand limits are different, we conclude that $\lim_{x\to 1} f(x)$ does not exist.

Recall that a function $f(x)$ is continuous at a point $x = a$ if three conditions are satisfied at the same time:

- $\lim_{x\to a} f(x)$ exists;
- $f(a)$ is defined (i.e. a is in the domain of $f(x)$);
- $\lim_{x\to a} f(x) = f(a)$.

The failure of any condition results in a discontinuity of $f(x)$ at the number $x = a$.