Quiz 5 Solutions

Problem 1-3. The length of a rectangle is increasing at a rate of 5 cm/s and its width is increasing at a rate of 3 cm/s. In the following questions, you will find how fast is the area of the rectangle increasing when the length is 20 cm and the width is 10 cm.

1. If we assume the rectangle's area is A cm^2, length is x cm, width is y cm and t is the time (in seconds). Use A, x, y, t or their derivatives to express "the length is increasing at a rate of 5 cm/s and the width is increasing at a rate of 3 cm/s. "

   Solution.
   $$\frac{dx}{dt} = 5 \quad \text{and} \quad \frac{dy}{dt} = 3\,.$$

2. *(2 points)*

   (a) Use A, x, y, and t, or their derivatives, to write an expression for the rate of increase of the area of the rectangle. (By definition, no computation needed.)

   (b) Can you find an equation that relates A, x and y?

   Solution.

   (a) $\frac{dA}{dt}$

   (b) $A = xy$

3. *(2 points)* (Continuation of Problem 2.) Find how fast is the area of the rectangle increasing when the length is 20 cm and the width is 10 cm.

   Solution.
   $$\frac{dA}{dt} = \frac{d\,(xy)}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}\,.$$
   Plug in $x = 20$, $y = 10$, $\frac{dx}{dt} = 5$ and $\frac{dy}{dt} = 3$ we can find
   $$\frac{dA}{dt} = 5 \times 10 + 20 \times 3 = 110\,.$$

4. *(2 points)* Find all the critical numbers of the function $f(x) = \sqrt{x}e^{-2x}$.

   Solution.
   $$f'(x) = \frac{1}{2\sqrt{x}}\,e^{-2x} - 2\sqrt{x}e^{-2x}$$
   If $f'(x) = 0$, $\frac{1}{2\sqrt{x}}\,e^{-2x} - 2\sqrt{x}e^{-2x} = 0$. Because e^{-2x} is never equal to zero,
   $$\frac{1}{2\sqrt{x}} - 2\sqrt{x} = 0 \Rightarrow \frac{1}{2\sqrt{x}} = 2\sqrt{x} \Rightarrow 1 = 4x$$
   therefore $x = \frac{1}{4}$.

   On the other hand, $f'(x)$ is not defined when $x = 0$. Therefore the critical numbers are 0 and $\frac{1}{4}$.

5. *(2 points)* On what intervals is the graph of the function $f(x) = x^4 - 3x^2$ concave up?

Solution.

$$f'(x) = 4x^3 - 6x \Rightarrow f''(x) = 12x^2 - 6$$

The graph of f is concave up on an interval if and only if $f''(x) > 0$ on that interval.

$$12x^2 - 6 > 0 \Rightarrow x^2 > \frac{6}{12} = \frac{1}{2} \Rightarrow x > \sqrt{\frac{1}{2}} \text{ or } x < -\sqrt{\frac{1}{2}}.$$

The intervals are $(-\infty, -\sqrt{\frac{1}{2}})$ and $(\sqrt{\frac{1}{2}}, \infty)$.