

# A NOTE ON SINGULAR MODULI SPACES OF SHEAVES ON K3 SURFACES

ZIYU ZHANG

ABSTRACT. This paper studies deformations and birational maps between singular moduli spaces of torsion free semistable sheaves with 2-divisible Mukai vectors on K3 surfaces. It is showed that when the greatest common divisor of the rank and the first Chern class is 2, two such moduli spaces of the same dimension can be connected by deformations and birational maps.

## CONTENTS

1. Introduction	1
2. Deformation Equivalence	3
3. Birational Equivalence	9
4. Rank 0 Case	14
References	19

## 1. INTRODUCTION

Moduli spaces of semistable sheaves have been studied for a long time. Let  $X$  be a K3 surface, and  $H \in \text{Pic}(X)$  be an ample line bundle. Let  $v \in H^{\text{even}}(X, \mathbb{Z})$  be a fixed Mukai vector. Then there is a moduli space  $\mathfrak{M}_{X,H}(v)$  which parameterizes S-equivalence classes of semistable sheaves with respect to the polarization  $H$ , whose Mukai vectors are  $v$ . These moduli spaces were first constructed by Gieseker [Gie77] and Maruyama [Mar77, Mar78], and then studied by many other people.

When the polarization  $H$  is generic and the Mukai vector  $v$  is primitive, namely, the greatest common divisor of all components of  $v$  is 1, every semistable sheaf must be stable. In this case, Mukai [Muk84, Corollary 0.2] proved that the moduli space  $\mathfrak{M}_{X,H}(v)$  is a smooth irreducible holomorphic symplectic manifold.

It is an interesting problem to study the relation among these smooth moduli spaces. There is a very nice theorem on irreducible holomorphic symplectic manifolds due

---

2010 *Mathematics Subject Classification.* 14D20.

*Key words and phrases.* Moduli spaces of sheaves, K3 surfaces, exceptional bundles.

to Huybrechts [Huy03, Theorem 2.5], which states that two birational irreducible holomorphic symplectic manifolds are always deformation equivalent. Applying this theorem, Yoshioka proved that [Yos01b, Theorem 8.1], if two such smooth moduli spaces have the same dimension, then they are deformation equivalent. This is really a nice result. However, it is not good news in the study of irreducible holomorphic symplectic manifolds. The reason is, although we have lots of choices for the underlying K3 surface, the generic polarization and the primitive Mukai vector, the resulting moduli spaces provide only one deformation type of holomorphic symplectic manifolds in every even dimension.

In this paper, we are trying to generalize Yoshioka's result to the case of 2-divisible Mukai vectors, which are Mukai vectors whose greatest common divisor among all components is 2. More precisely, we will prove the following theorem:

**Theorem 1.1.** *For  $i = 1$  or  $2$ , assume  $X_i$  is a projective K3 surface,  $v_i = (r_i, c_i, a_i)$  is a primitive Mukai vector with  $r_i > 0$ .  $H_i$  is a polarization on  $X_i$  which is generic with respect to the Mukai vector  $2v_i$ . Assume further that  $r_i$  and  $c_i$  are coprime. If  $\dim \mathfrak{M}_{X_1, H_1}(2v_1) = \dim \mathfrak{M}_{X_2, H_2}(2v_2)$ , then  $\mathfrak{M}_{X_1, H_1}(2v_1)$  and  $\mathfrak{M}_{X_2, H_2}(2v_2)$  can be connected by a sequence of deformations and birational maps.*

Note that  $(r_i, c_i) = (1, 0)$  is allowed by the assumption of the theorem. In particular, the case of  $(r_i, c_i, a_i) = (1, 0, -1)$  is exactly the moduli space studied by O'Grady in [O'G99], which admits a symplectic resolution.

The proof contains two main steps. In the first step, we will show that, for two such moduli spaces  $\mathfrak{M}_{X_1, H_1}(2v_1)$  and  $\mathfrak{M}_{X_2, H_2}(2v_2)$ , if they have the same dimension and  $r_1 = r_2$ , then they are deformation equivalent. The idea of this part of proof is to use the deformations of polarized K3 surfaces to deform both of the given moduli spaces to a third moduli space of sheaves on an elliptic K3 surface with a rank 2 Picard lattice. So that we know all moduli spaces in the same dimension with the same rank parameter are deformation equivalent. This part of proof was motivated by a similar argument in [HL97]. In the second step, we fix the dimension of the moduli spaces and let the rank vary. For every possible value of the rank component of the Mukai vector which is at least 8, we find one particular moduli space of sheaves on a K3 surface with rank 1 Picard lattice, and prove it is birational to a certain moduli space of sheaves of rank 2. The birational map between these two moduli spaces is established via extensions by exceptional bundles, which were introduced in [Yos99b]. Combining the two steps, any two moduli spaces of sheaves of rank not equal to 4, as stated in theorem 1.1, can be connected in at most three steps, namely a deformation followed by a birational map, then by another deformation. For a technical reason, we have to deal with the case of rank 4 separately. To connect these remaining moduli spaces, we just need to realize that, in every dimension we can find a moduli space of sheaves of rank 4, which is birational to another moduli space of rank 14. So now all moduli spaces of the same dimension are connected by deformations and birational maps.

However, due to the lack of a version of Huybrechts's theorem [Huy03, Theorem 2.5] in the singular case, here we cannot get rid of the birational maps and get a theorem as nice as Yoshioka's [Yos01b, Theorem 8.1]. However, we hope a similar result of deformation equivalence of singular moduli spaces in the same dimension is still true for arbitrary non-primitive Mukai vectors. So we can formulate the following conjecture:

**Conjecture 1.2.** *Let  $m$  be a positive integer. For  $i = 1$  or  $2$ , assume  $X_i$  is a projective K3 surface,  $v_i = (r_i, c_i, a_i)$  is a primitive positive Mukai vector, and  $H_i$  is a polarization which is generic with respect to the Mukai vector  $mv_i$ . If  $\dim M_{X_1, H_1}(mv_1) = \dim M_{X_2, H_2}(mv_2)$ , then  $M_{X_1, H_1}(mv_1)$  and  $M_{X_2, H_2}(mv_2)$  are deformation equivalent.*

This paper is organized as follows: the two steps in the proof of theorem 1.1 will be done in sections 2 and 3. In the proof of the deformation equivalence we will need a result of local finiteness of walls in the ample cone (lemma 2.5), whose proof will be given in the end of section 2. In section 3, for every  $r \geq 3$  we will find a moduli space of sheaves of rank  $2r$ , and prove it is birational to a moduli space of sheaves of rank 2. The remaining case for  $r = 2$  will be treated at the end of the section (propositions 3.9). We will omit the proofs because they will be identical with that of the proposition 3.1. These two sections finish the proof of Theorem 1.1. Finally, in section 4, we will prove that certain moduli spaces of pure torsion sheaves with 2-divisible Mukai vectors are birational to moduli spaces of sheaves of positive ranks.

After the first version of this paper was completed and posted on arXiv, conjecture 1.2 was proved in full generality by [Per11]. The author of [Per11] followed the argument used in the discussion for 10 dimensional moduli spaces in [PR10]. One of the main tools used in [Per11] is an isomorphism between two moduli spaces given by a Fourier-Mukai transform, which was proved in [Yos01a, Theorem 3.18]. However, the technique used in the present paper is very different from that in [Per11] and is more elementary.

**Acknowledgements:** I would like to express my deep gratitude to my PhD advisor Jun Li, for all his support and encouragement. I would also like to thank Kōta Yoshioka for kindly answering my questions and pointing out references, and thank Jason Lo for helpful discussions. I also appreciate the help of Daniel Huybrechts and Manfred Lehn in the final stage of this work. Moreover, I would like to thank the referee for carefully reading the previous version of the manuscript and suggesting various improvements, which in particular leads to a few simplifications in section 3.

## 2. DEFORMATION EQUIVALENCE

In this section, we will show that any two moduli spaces of sheaves on K3 surfaces with the same dimension, rank and divisibility are deformation equivalent to each other. This will be obtained by deforming the two moduli spaces to a third moduli space of sheaves of the same rank over an elliptic K3 surface. More precisely, we will show:

**Proposition 2.1.** *Let  $X$  be a K3 surface,  $v = (r, c, a) \in H^{\text{even}}(X)$  be a Mukai vector with  $\gcd(r, c) = 1$ ,  $H$  be a generic polarization. Let  $X^e$  be an elliptic K3 surface with  $\text{Pic}(X^e) = \mathbb{Z}[\sigma] \oplus \mathbb{Z}[f]$  where  $\sigma$  is the class of a section of the elliptic fibration and  $f$  is the fiber class. The moduli space  $\mathfrak{M}_{X,H}(2v)$  is deformation equivalent to a certain moduli space of sheaves  $\mathfrak{M}_{X^e, \sigma+lf}(2v')$  on  $X^e$ , where  $v' = (r, \sigma + lf, a')$  for some  $l$  and  $a'$ . Moreover, the polarization  $\sigma + lf$  is  $v'$ -suitable.*

We recall the notion of a suitable polarization. Note that for such an elliptic K3 surface, the ample cone is 2-dimensional, and the ray generated by the fiber class  $f$  is a boundary of the ample cone, since for any pair of positive integers  $a$  and  $b$  satisfying  $2a < b$ , the class  $a\sigma + bf$  is ample by Nakai-Moishezon criterion. Therefore for any fixed Mukai vector  $v = (r, c, a)$ , there is exactly one chamber in the ample cone, which has this ray (generated by  $f$ ) as a boundary. We say an ample class is *suitable* with respect to  $v$  if it is contained in the interior of this chamber. (See for instance Definition 5.3.1 and Remark 5.3.6 in [HL97].) We also recall the following proposition on suitable polarizations.

**Lemma 2.2.** [HL97, Remark 5.3.6] *Under the same notations as above, the class  $\sigma + mf$  is suitable if  $m \geq 3 + r^2\Delta(v)/8$ , where  $\Delta(v) = 2r^2 + \langle v^2 \rangle$  is the discriminant of the Mukai vector  $v$ .  $\square$*

Before proving proposition 2.1, we have to consider moduli spaces of polarized K3 surfaces. Much of this has been summarized in [HL97].

Let  $d$  be a positive number, we consider all K3 surfaces  $X$  with ample primitive line bundles  $L$  satisfying  $c_1^2(L) = 2d$ . Then there is a quasi-projective scheme  $\mathcal{K}_d$ , which is a coarse moduli space of all such pairs  $(X, L)$ .

This moduli space can be constructed by GIT. More precisely, it is a  $\text{PGL}(N)$  quotient of  $\mathcal{H}_d$  which is an open subset of a certain Hilbert scheme. The universal family over the Hilbert scheme provides a universal family of polarized K3 surfaces over  $\mathcal{H}_d$ . For every point  $t \in \mathcal{H}_d$ , the fiber of this universal family  $(\mathcal{X}_t, \mathcal{L}_t)$  is exactly the pair corresponding to the image of  $t$  in  $\mathcal{K}_d$ .

Furthermore, we know that both  $\mathcal{K}_d$  and  $\mathcal{H}_d$  are irreducible. Therefore, any two primitively polarized K3 surfaces  $(X_1, L_1)$  and  $(X_2, L_2)$  with  $L_1^2 = L_2^2$  are deformation equivalent to each other.

Another fact which will be used later is: for a general polarized K3 surface  $(X, L) \in \mathcal{K}_d$ , we have  $\text{Pic}(X) = \mathbb{Z}[H]$ . In other words, in the moduli space  $\mathcal{K}_d$  of primitively polarized K3 surfaces, away from a countable union of Zariski closed subsets, the Picard number  $\rho(X) = 1$ . However, the countable union of polarized K3 surfaces  $(X, H) \in \mathcal{K}_d$  with  $\rho(X) \geq 2$  is also dense in  $\mathcal{K}_d$ .

We will need the following proposition:

**Proposition 2.3.** *Let  $r$  and  $k$  be positive integers and  $a$  be an arbitrary integer. Then there is a relative moduli space  $\varphi : \mathfrak{M} \rightarrow \mathcal{H}_d$  of semistable sheaves, such*

that for every  $t \in \mathcal{H}$ ,  $\varphi^{-1}(t)$  is isomorphic to the moduli space of semistable sheaves  $\mathfrak{M}_{\mathcal{X}_t, \mathcal{L}_t}(2v_t)$  where  $v_t = (r, k\mathcal{L}_t, a)$ .

*Proof.* The proposition is a special case of the existence theorem of relative moduli spaces of semistable sheaves ([HL97, Theorem 4.3.7]).  $\square$

We will also need the following simple observation of Yoshioka on the preservation of Gieseker stability under tensor product with a line bundle.

**Lemma 2.4.** [Yos01b, Lemma 1.1] *Let  $(X, H)$  be a polarized K3 surface,  $v$  be an arbitrary Mukai vector with positive rank and  $H$  is a generic polarization with respect to  $v$ . Then  $F$  is  $H$ -stable (resp. semistable) if and only if  $F \otimes L$  is  $H$ -stable (resp. semistable). In other words, the moduli spaces  $\mathfrak{M}_{X, H}(v)$  and  $\mathfrak{M}_{X, H}(v \cdot \text{ch}(L))$  are isomorphic.*  $\square$

Now we are ready to prove proposition 2.1. The proof will be parallel to the proof of Theorem 6.2.5 in [HL97].

*Proof of Proposition 2.1.* The proof is presented in 4 steps.

*Step 1.* Without loss of generality, we can assume that  $c$  is an ample class.

In fact, if  $c$  is not ample, we can apply lemma 2.4 to get a new moduli space isomorphic to the original one. More precisely, for every semistable sheaf  $F$  with Mukai vector  $2v = 2(r, c, a)$ , we consider a new sheaf  $F \otimes H^{\otimes m}$ . We have that

$$v(F \otimes H^{\otimes m}) = v(F) \cdot \text{ch}(H)^{\otimes m} = 2(r, c + rmH, a + cmH + \frac{rm^2}{2}H^2).$$

We denote the new Mukai vector  $(r, c + rmH, a + cmH + \frac{rm^2}{2}H^2)$  by  $v'$ . By lemma 2.4 we have an isomorphism between  $\mathfrak{M}_{X, H}(2v)$  and  $\mathfrak{M}_{X, H}(2v')$ . Since  $H$  is ample, we know that  $c + rmH$  is also an ample class when  $m$  is sufficiently large. Therefore we can replace  $\mathfrak{M}_{X, H}(2v)$  by  $\mathfrak{M}_{X, H}(2v')$ .

*Step 2.* Without loss of generality, we can assume that  $\rho(X) \geq 2$ .

If not, then  $\text{Pic}(X) = \mathbb{Z}[H]$ . Let  $c = kH$  where  $k > 0$ . We can consider the relative moduli space  $\varphi : \mathfrak{M} \rightarrow \mathcal{H}_d$  in proposition 2.3. Our moduli space  $\mathfrak{M}_{X, H}(2r, 2kH, 2a)$  is a fiber of this relative moduli space over the point  $(X, H)$ . Since polarized K3 surfaces with Picard number at least two are dense in  $\mathcal{H}_d$ , a priori, we can choose any polarized K3 surface  $(X_t, H_t)$  near  $(X, H)$  in  $\mathcal{H}_d$ , with  $\rho(X_t) \geq 2$ , so that the original moduli space  $\mathfrak{M}_{X, H}(2r, 2kH, 2a)$  is deformation equivalent to the new moduli space  $\mathfrak{M}_{X_t, H_t}(2r, 2kH_t, 2a)$ .

However, we still have to make sure that  $H_t$  is a generic polarization on  $X$  with respect to the Mukai vector  $(2r, 2kH_t, 2a)$ . We need the following lemma:

**Lemma 2.5.** *In a sufficiently small open neighborhood of  $(X, H)$  in  $\mathcal{H}_d$ , there are at most finitely many hypersurfaces, such that for every point  $(X_t, H_t)$  not on those*

hypersurfaces, the polarization  $H_t$  is generic with respect to the corresponding Mukai vector  $2(r, kH_t, a)$ .

We will postpone the proof of this lemma to the end of this section.

By virtue of the above lemma, we can always find a polarized K3 surface  $(X_t, H_t)$  with  $\rho(X_t) \geq 2$ , which is deformation equivalent to the original polarized K3 surface  $(X, H)$ .

*Step 3.* We can even assume that  $c = H$  and  $H^2 \gg 0$ .

In fact, since  $H$  is in an open chamber and  $\rho(X) \geq 2$ , we can always pick another primitive ample line bundle  $H'$  in the same open chamber as  $H$ , which is linearly independent with  $c$ . Therefore  $\mathfrak{M}_{X,H}(2v)$  is canonically the same as  $\mathfrak{M}_{X,H'}(2v)$ . We can now apply lemma 2.4 and twist the sheaves in the moduli space  $\mathfrak{M}_{X,H'}(2v)$  by  $H'$  for a few times to get an isomorphic moduli space. More precisely, if  $v(F) = 2(r, c, a)$ , then

$$v(F \otimes H'^{\otimes m}) = v(F) \cdot \text{ch}(H')^{\otimes m} = 2(r, c + rmH', a + cmH' + \frac{rm^2}{2}H'^2).$$

We denote the Mukai vector  $(r, c + rmH', a + cmH' + \frac{rm^2}{2}H'^2)$  by  $v'$ . By lemma 2.4,  $\mathfrak{M}_{X,H'}(2v)$  is isomorphic to  $\mathfrak{M}_{X,H'}(2v')$ . Note that when  $m$  is sufficiently large,  $c + rmH'$  is very close to  $H'$ , therefore is also ample and in the same chamber as  $H'$ , or  $H$ . So the moduli space  $\mathfrak{M}_{X,H'}(2v')$  is isomorphic to  $\mathfrak{M}_{X,c+rmH'}(2v')$ . With the additional assumption that  $r$  and  $c$  are coprime, we know that  $c + rmH'$  is primitive. Increase  $m$  we can make  $(c + rmH')^2 \gg 0$ .

*Step 4.* We can finish the proof based on the assumptions made in the previous step.

Now We consider an elliptic K3 surface  $X^e$  with  $\text{Pic}(X^e) = \mathbb{Z}[\sigma] \oplus \mathbb{Z}[f]$  where  $\sigma$  is the class of a section and  $f$  is the fiber class, as stated in the proposition. We take  $l = H^2 + 2$ . Then it is clear that  $H^2 = (\sigma + lf)^2$ . By the previous step we can assume  $H^2 \gg 0$ , therefore we have  $l \gg 0$ . Since the discriminant  $\Delta(2v)$  is the same as  $\Delta(2(r, \sigma + lf, a))$ , the assumption of lemma 2.2 holds when  $l > 3 + 2r^2\Delta(v)$ , therefore we know  $\sigma + lf$  is ample and suitable with respect to the Mukai vector  $2(r, \sigma + lf, a)$ . Hence both of the polarized K3 surfaces  $(X, H)$  and  $(X^e, \sigma + lf)$  lie in  $\mathcal{H}_d$ . Now we apply proposition 2.3 again with the assumption  $k = 1$  to conclude that  $\mathfrak{M}_{X,H}(2(r, c, a))$  is deformation equivalent to  $\mathfrak{M}_{X^e, \sigma + lf}(2(r, \sigma + lf, a))$ . □

From proposition 2.1 we can prove the following corollary:

**Corollary 2.6.** *For  $i = 1, 2$ , let  $X_i$  be a K3 surface,  $v_i = (r_i, c_i, a_i) \in H^{\text{even}}(X_i, \mathbb{Z})$  be a primitive Mukai vector with  $\gcd(r_i, c_i) = 1$  and  $H_i$  be a generic ample line bundle on  $X_i$  with respect to  $v_i$ . If  $r_1 = r_2 = r$  and  $\langle v_1^2 \rangle = \langle v_2^2 \rangle$ , then moduli spaces of semistable sheaves  $\mathfrak{M}_{X_1, H_1}(2v_1)$  is deformation equivalent to  $\mathfrak{M}_{X_2, H_2}(2v_2)$ .*

*Proof.* By proposition 2.1, there exists an elliptic K3 surface  $X^e$ , such that, for  $i = 1$  and 2, there exists a positive integer  $l_i$ , such that  $\mathfrak{M}_{X^e, H_i}(2v_i)$  is deformation equivalent to  $\mathfrak{M}_{X^e, \sigma + l_i f}(2(r, \sigma + l_i f, a_i))$ .

Now we claim the two moduli spaces

$$\mathfrak{M}_1 = \mathfrak{M}_{X^e, \sigma + l_1 f}(2(r, \sigma + l_1 f, a_1)) \quad \text{and} \quad \mathfrak{M}_2 = \mathfrak{M}_{X^e, \sigma + l_2 f}(2(r, \sigma + l_2 f, a_2))$$

are isomorphic. For simplicity we write  $w_i = 2(r, \sigma + l_i f, a_i)$  for  $i = 1, 2$ .

In fact, from the given condition we know the two moduli spaces have the same dimension, therefore  $\langle w_1^2 \rangle = \langle w_2^2 \rangle$ , that is,  $(\sigma + l_1 f)^2 - 2ra_1 = (\sigma + l_2 f)^2 - 2ra_2$ . This implies  $2r(a_2 - a_1) = 2(l_2 - l_1)\sigma \cdot f = 2(l_2 - l_1)r$ , so  $r$  divides  $l_2 - l_1$ . Let  $l_2 - l_1 = r \cdot h$ . If  $h = 0$  then we are left nothing to prove. Without loss of generality, we assume  $h > 0$ .

By lemma 2.2 we know that  $\sigma + l_1 f$  is  $w_1$ -suitable. Since  $l_1 < l_2$ ,  $\sigma + l_2 f$  is also  $w_1$ -suitable (in particular,  $w_1$ -generic), and therefore  $\mathfrak{M}_1$  is the same as  $\mathfrak{M}_{X^e, \sigma + l_2 f}(w_1)$ . Now we apply lemma 2.4 with  $L = \mathcal{O}(hf)$  on this moduli space. A simple calculation shows that  $w_1 \otimes \text{ch}(L) = w_2$ , therefore  $\mathfrak{M}_1$  is further isomorphic to  $\mathfrak{M}_{X^e, \sigma + l_2 f}(w_2) = \mathfrak{M}_2$ .

This proves the two original moduli spaces are deformation equivalent to each other.  $\square$

To conclude this section we prove the Lemma 2.5. The idea is similar to Lemma 4.C.2 in [HL97]. For the proof we need to state the following lemma:

**Lemma 2.7.** [HL97, Theorem 4.C.3] *Let  $H$  be an ample line bundle, and  $F$  be a semistable coherent sheaf with Mukai vector  $v(F) = (r, c, a)$ . Let  $F'$  be a subsheaf of  $F$  with Mukai vector  $v(F') = (r', c', a')$ , where  $0 < r' < r$ , such that the reduced Hilbert polynomials  $p(F, m) = p(F', m)$ . Let  $\Delta = 2r^2 + c^2 - 2ra$  be the discriminant of  $F$ . Then we have*

- (1)  $ra' - r'a = 0$ ;
- (2) let  $\xi = rc' - r'c$ , then  $\xi \cdot H = 0$ ;
- (3)  $-\frac{r^2}{4}\Delta \leq \xi^2 \leq 0$  and  $\xi^2 = 0$  if and only if  $\xi = 0$ .  $\square$

Now we can begin to prove lemma 2.5.

*Proof of Lemma 2.5.* We consider the walls in  $H^2(X, \mathbb{R})$ . Let  $\sigma$  be a generator of  $H^{2,0}(X)$ , then  $\bar{\sigma}$  is a generator of  $H^{0,2}(X)$ . Let

$$e_1 = \frac{\sigma + \bar{\sigma}}{\sqrt{(\sigma + \bar{\sigma})^2}}$$

and

$$e_2 = \frac{\sigma - \bar{\sigma}}{\sqrt{-(\sigma - \bar{\sigma})^2}}.$$

Then both  $e_1$  and  $e_2$  are unit vectors in  $H^2(X, \mathbb{R})$ . Furthermore, since the signature of Poincare pairing on  $H^2(X, \mathbb{R})$  is  $(3, h^{1,1} - 3)$ , we see that  $\{e_1, e_2, H\}$  is an orthogonal basis of a three dimensional subspace  $V$  of  $H^2(X, \mathbb{R})$ . Therefore, for every class  $u \in H^2(X, \mathbb{R})$ , we can decompose it into  $u = a_1 e_1 + a_2 e_2 + a_0 H + u_0$  where  $a_0, a_1, a_2 \in \mathbb{R}$  and  $u_0 \in V^\perp$ . Besides the Poincare pairing, we can define an Euclidean pairing on  $H^2(X, \mathbb{R})$ , by  $\|u\|^2 = \sqrt{a_1^2 + a_2^2 + a_0^2 H^2 - u_0^2}$ .

For every pair  $(X', H') \in \mathcal{H}_d$ , we can produce  $e'_1, e'_2$  in the same way as  $e_1, e_2$ . In a sufficiently small open neighborhood of the pair  $(X, H)$  in  $\mathcal{H}_d$ , every base manifold  $X'$  is a small deformation of  $X$ , whose  $H^{2,0}$  and  $H^{0,2}$  are very closed to  $H^{2,0}(X)$  and  $H^{0,2}(X)$ . More precisely, we can require  $\|e'_1 - e_1\| < \varepsilon$  and  $\|e'_2 - e_2\| < \varepsilon$ .

Since  $e'_1$  and  $e'_2$  are both perpendicular to  $H$ , let  $e'_1 = \lambda_1 e_1 + \delta_1 e_2 + h_1$  where  $h_1 \in V^\perp$  and  $e'_2 = \delta_2 e_1 + \lambda_2 e_2 + h_2$  where  $h_2 \in V^\perp$ . Then  $\|e'_1 - e_1\| < \varepsilon$  implies  $(1 - \lambda_1)^2 + \delta_1^2 + \|h_1\|^2 < \varepsilon^2$ . So we can conclude  $|1 - \lambda_1| < \varepsilon$ ,  $|\delta_1| < \varepsilon$ ,  $\|h_1\| < \varepsilon$ . Similarly, from  $\|e'_2 - e_2\| < \varepsilon$ , we can deduce  $|1 - \lambda_2| < \varepsilon$ ,  $|\delta_2| < \varepsilon$ ,  $\|h_2\| < \varepsilon$ .

By lemma 2.7, we know that there exists a positive constant  $\Delta$ , such that for every  $\xi \in H^{1,1}(X')$  which produces a wall in  $H^2(X', \mathbb{R}) = H^2(X, \mathbb{R})$  for some pair  $(X', H')$  in the small neighborhood of  $(X, H)$ , we have  $-\Delta < \xi^2 \leq 0$ . Since  $\xi \perp H$ , we can decompose  $\xi$  as  $\xi = a_1 e_1 + a_2 e_2 + \xi_0$  where  $\xi_0 \in V^\perp$ . We want to show the Euclidean norm  $\|\xi\|^2 = a_1^2 + a_2^2 - \xi_0^2$  is bounded.

Note that  $\xi \perp e'_1$  and  $\xi \perp e'_2$ , that is,

$$\begin{aligned}\lambda_1 a_1 + \delta_1 a_2 + h_1 \cdot \xi_0 &= 0 \\ \delta_2 a_1 + \lambda_2 a_2 + h_2 \cdot \xi_0 &= 0\end{aligned}$$

Therefore we have

$$\begin{aligned}(\lambda_1 a_1 + \delta_1 a_2)^2 &= (h_1 \cdot \xi_0)^2 \leq \|h_1\|^2 \|\xi_0\|^2 \\ (\delta_2 a_1 + \lambda_2 a_2)^2 &= (h_2 \cdot \xi_0)^2 \leq \|h_2\|^2 \|\xi_0\|^2\end{aligned}$$

We add the two inequalities. The left hand side is

$$\begin{aligned}& (\lambda_1^2 + \delta_2^2) a_1^2 + (\delta_1^2 + \lambda_2^2) a_2^2 + 2(\lambda_1 \delta_1 + \delta_2 \lambda_2) a_1 a_2 \\ & > (1 - \varepsilon)^2 (a_1^2 + a_2^2) - 4\varepsilon(1 + \varepsilon) a_1 a_2 \\ & > (1 - \varepsilon)^2 (a_1^2 + a_2^2) - 2\varepsilon(1 + \varepsilon) (a_1^2 + a_2^2) \\ & = (1 - 4\varepsilon - \varepsilon^2) (a_1^2 + a_2^2).\end{aligned}$$

So now we have

$$(1 - 4\varepsilon - \varepsilon^2) (a_1^2 + a_2^2) < 2\varepsilon^2 \|\xi_0\|^2.$$

Therefore we have

$$-\Delta < a_1^2 + a_2^2 - \|\xi_0\|^2 < \left( \frac{2\varepsilon^2}{1 - 4\varepsilon - \varepsilon^2} - 1 \right) \|\xi_0\|^2.$$

So

$$\|\xi_0\|^2 < \frac{\Delta}{1 - \frac{2\varepsilon^2}{1 - 4\varepsilon - \varepsilon^2}}.$$



Therefore

$$\|\xi\|^2 = a_1^2 + a_2^2 + \|\xi_0\|^2 \leq 2\|\xi_0\|^2 < \frac{2\Delta}{1 - \frac{2\varepsilon^2}{1-4\varepsilon-\varepsilon^2}}$$

which is bounded regardless of the underlying K3 surface  $X'$ .

Therefore, in a small open neighborhood of  $(X, H)$  in  $\mathcal{H}_d$ , there are at most finitely many hypersurfaces, on which the polarization is not generic. Since we have assumed that  $H$  is in generic on  $X$ , we can always shrink the open neighborhood of  $(X, H)$  in  $\mathcal{H}_d$ , so that it misses all these hypersurfaces. In other words, for every point  $(X_t, H_t)$  in this open neighborhood,  $H'$  is generic with respect to the Mukai vector  $2(r, kH_t, a)$ .  $\square$

### 3. BIRATIONAL EQUIVALENCE

In this section, we will establish the birational equivalence between some singular moduli spaces of semistable sheaves, so that together with the deformation result we obtained in the previous section, for any two moduli spaces of sheaves in the same dimension, both with 2-divisible Mukai vectors, we can connect them by a series of deformations and birational maps. For this purpose, in this section we always fix the underlying K3 surface, which is a projective K3 surface with Picard number 1. Roughly speaking, we will prove that for every value of  $r$ , there is a moduli space of rank  $2r$  sheaves, which is birational to a moduli space of rank 2 sheaves (in case  $r = 2$ , birational to a moduli space of sheaves of higher rank). The method we are using here in proving the birationality is mainly the technique of exceptional bundles introduced by Yoshioka in [Yos99b]. Similar to [Yos99b], the proof here involves some delicate analysis of slopes of sheaves.

The main theorem of this section is:

**Proposition 3.1.** *For any given  $r \geq 3$  and  $s \geq 1$ , let  $X$  be a projective K3 surface with  $\text{Pic}(X) = \mathbb{Z}[H]$  where  $H$  is an ample line bundle with  $H^2 = 2s(r-1)^2 - 2r$ . Consider Mukai vectors*

$$\begin{aligned} v_0 &= (r-1, (r-2)H, (r^3 - 5r^2 + 8r - 4)s + (-r^2 + 3r - 1)), \\ v_1 &= (r, (r-1)H, (r^3 - 4r^2 + 6r - 4)s + (-r^2 + 2r - 1)), \\ v_2 &= (1, H, (r^2 - 2r)s - r). \end{aligned}$$

*Then the moduli spaces of semi-stable sheaves  $\mathfrak{M}_{X,H}(2v_1)$  is birational equivalent to  $\mathfrak{M}_{X,H}(2v_2)$ .*

Simple calculation shows that  $\langle v_1^2 \rangle = \langle v_2^2 \rangle = 2s$ , therefore moduli spaces  $\mathfrak{M}_{X,H}(2v_1)$  and  $\mathfrak{M}_{X,H}(2v_2)$  have the same dimension. We also have  $\langle v_0, v_1 \rangle = -1$  and  $\langle v_0, v_2 \rangle = 1$ . Furthermore, we have  $\langle v_0^2 \rangle = -2$ . Due to the following lemma of Yoshioka, the technique of exceptional bundles applies.

**Lemma 3.2.** [Yos99b, Theorem 3.6] *The moduli space  $\mathfrak{M}_{X,H}(v_0)$  consists of one point which corresponds to a  $\mu$ -stable locally free sheaf  $E$ . We call such a locally free sheaf an exceptional bundle.*  $\square$

We should also note that, in proposition 3.1, we require  $r \geq 3$ . The reason is that, the formula for  $H^2$  in the setup of the proposition doesn't produce a positive number for small values of  $s$ . However, by carefully choosing the numerical data in these two cases, the proof will work in exactly the same way for the remaining case. We will state a theorem for this remaining case in the end of this section and omit its proof. Note that the same problem also happened in the proof of [Yos99b, Theorem 3.8], and was dealt with in the same way.

Before we prove proposition 3.1, we want to show that a generic point in the moduli spaces  $\mathfrak{M}_{X,H}(2v_1)$  or  $\mathfrak{M}_{X,H}(2v_2)$  is represented by a  $\mu$ -stable locally free sheaf. We recall the following lemma due to Yoshioka:

**Lemma 3.3.** [Yos99a, Remark 2.2][Yos99b, Lemma 4.4, Remark 4.3] *Let  $(X, H)$  be a generic polarized K3 surface. Let  $v = (lr, l\xi, a)$  be a Mukai vector, such that  $r$  and  $\xi$  are coprime. Then there is at least one  $\mu$ -stable sheaf with Mukai vector  $v$  unless the following two conditions simultaneously hold:*

- $\frac{\xi^2 + 2}{2r}$  is an integer;
- $\langle v^2 \rangle < 2l^2$ .  $\square$

We apply the above lemma in our situation. Note that when  $v = 2v_1$  or  $2v_2$ , we always have

$$\langle v^2 \rangle = 8s \geq 8 = 2l^2.$$

The second condition in the above lemma fails, both of the moduli spaces  $\mathfrak{M}_{X,H}(2v_1)$  and  $\mathfrak{M}_{X,H}(2v_2)$  contain at least one  $\mu$ -stable sheaf. Take into consideration that the  $\mu$ -stability is an open condition, we know that each of the two moduli spaces has an open subscheme, which parametrizes  $\mu$ -stable sheaves. Moreover, the following lemma due to Yoshioka shows that a generic point in the  $\mu$ -stable locus is represented by a locally free sheaf:

**Lemma 3.4.** [Yos01b, Remark 3.2] *Let  $r$  be the rank component of the Mukai vector  $v$ . Then the complement of the locus of locally free sheaves in the moduli of  $\mu$ -stable sheaves  $\mathfrak{M}_{X,H}^{\mu-st}(v)$  has codimension  $r - 1$ .*  $\square$

In our situation, the rank of the sheaf is at least 2. So we conclude that

**Corollary 3.5.** *There are open subschemes  $U_1$  and  $U_2$  in  $\mathfrak{M}_{X,H}(2v_1)$  and  $\mathfrak{M}_{X,H}(2v_2)$  respectively, which parametrize all locally free  $\mu$ -stable sheaves with Mukai vector  $2v_1$  and  $2v_2$ .*  $\square$

First of all we need the following lemma, whose proof is very similar to [Yos99b, Lemma 2.1].

**Lemma 3.6.** *Let  $E$  be as given in lemma 3.2 For any  $\mu$ -stable locally free sheaf  $G$  in  $\mathfrak{M}_H(2v_2)$  and any non-trivial extension*

$$(1) \quad 0 \longrightarrow E \longrightarrow V \longrightarrow G \longrightarrow 0,$$

*then  $V$  is also a  $\mu$ -stable locally free sheaf.*

*Proof.* Since both  $E$  and  $G$  are locally free,  $V$  is also locally free. Assume  $V$  is not  $\mu$ -stable, then we can find a locally free  $\mu$ -stable subsheaf  $K$  such that

$$\frac{\deg K}{\text{rank } K} \geq \frac{\deg V}{\text{rank } V} = \frac{r}{r+1}.$$

In fact the inequality is strict because otherwise  $\text{rank } K$  will be a multiple of  $r+1$  which is a contradiction. So we have

$$\frac{\deg K}{\text{rank } K} \geq \frac{r}{r+1} > \frac{r-2}{r-1} = \frac{\deg E}{\text{rank } E}.$$

Since  $E$  is stable, there is no non-trivial map from  $K$  to  $E$ . So the composition  $K \longrightarrow V \longrightarrow G$  is non-trivial. Since  $G$  is  $\mu$ -stable, we have

$$\frac{\deg K}{\text{rank } K} \leq \frac{\deg G}{\text{rank } G} = 1.$$

So now we have

$$\frac{r}{r+1} < \frac{\deg K}{\text{rank } K} \leq 1.$$

If

$$\frac{r}{r+1} < \frac{\deg K}{\text{rank } K} < 1,$$

then

$$0 < \frac{\text{rank } K - \deg K}{\text{rank } K} < \frac{1}{r+1}.$$

Therefore we must have  $\text{rank } K > r+1$ , which is absurd. Therefore, we must have  $\frac{\deg K}{\text{rank } K} = 1$ . Now we consider the non-trivial map  $\varphi : K \longrightarrow G$ . It can be factored as  $K \rightarrow \text{Im } \varphi \hookrightarrow G$ . Since both  $K$  and  $G$  are  $\mu$ -stable, we have

$$1 = \frac{\deg K}{\text{rank } K} \leq \frac{\deg \text{Im } \varphi}{\text{rank } \text{Im } \varphi} \leq \frac{\deg G}{\text{rank } G} = 1.$$

Therefore both equalities hold. Due to the  $\mu$ -stability of  $K$ , the first equality implies that  $K = \text{Im } \varphi$ . In particular,  $\text{Im } \varphi$  is locally free. Due to the  $\mu$ -stability of  $G$ , the second equality implies that  $\text{rank } \text{Im } \varphi = \text{rank } G$  and  $\deg \text{Im } \varphi = \deg G$ . And since both  $\text{Im } \varphi$  and  $G$  are locally free, we can further conclude that  $\text{Im } \varphi = G$ . Hence  $K = G$ , which means the exact sequence (1) splits. It contradicts with the assumption that the extension is non-trivial.

So  $V$  is  $\mu$ -stable. □

By applying the same method, we can prove

**Lemma 3.7.** *For any  $\mu$ -stable locally free sheaf  $G$  in  $\mathfrak{M}_{X,H}(2v_2)$ , and any extension given by a 2-dimensional subspace of  $\text{Ext}^1(G, E)$*

$$0 \longrightarrow E^{\oplus 2} \longrightarrow F \longrightarrow G \longrightarrow 0,$$

*the sheaf  $F$  is a  $\mu$ -semistable locally free sheaf in  $\mathfrak{M}_{X,H}(2v_1)$ .*

*Proof.* We consider the push-out diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & E & \xlongequal{\quad} & E & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E^{\oplus 2} & \longrightarrow & F & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & E \oplus 0 & \longrightarrow & V & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Due to the fact that  $F$  corresponds to a two dimensional subspace of  $\text{Ext}^1(G, E)$ , the extension in the middle column

$$(2) \quad 0 \longrightarrow E \longrightarrow F \longrightarrow V \longrightarrow 0$$

is a non-trivial extension, where  $V$  itself is a locally free  $\mu$ -stable sheaf by lemma 3.6. We want to show that  $F$  is  $\mu$ -semistable. If not, let  $K$  be a locally free  $\mu$ -stable destabilizing sheaf of  $F$ . Then we have

$$\frac{\deg K}{\text{rank } K} > \frac{\deg F}{\text{rank } F} = \frac{r-1}{r}.$$

Since

$$\frac{\deg E}{\text{rank } E} = \frac{r-2}{r-1} < \frac{\deg K}{\text{rank } K},$$

we know there is no non-trivial map from  $K$  to  $E$ . Therefore the composition  $\varphi : K \longrightarrow F \longrightarrow V$  is a non-trivial map. By the  $\mu$ -stability of  $V$  we have

$$\frac{\deg K}{\text{rank } K} \leq \frac{\deg V}{\text{rank } V},$$

therefore now we have

$$\frac{r-1}{r} < \frac{\deg K}{\text{rank } K} \leq \frac{r}{r+1}.$$

If

$$\frac{r-1}{r} < \frac{\deg K}{\text{rank } K} < \frac{r}{r+1},$$

then we will have

$$r < \frac{\text{rank } K}{\text{rank } K - \text{deg } K} < r + 1,$$

which implies  $\text{rank } K > 2r$ . Contradiction!

Therefore we must have

$$\frac{\text{deg } K}{\text{rank } K} = \frac{r}{r + 1} = \frac{\text{deg } V}{\text{rank } V}.$$

The map  $\varphi : K \rightarrow V$  factors through  $\text{Im}\varphi$  as  $K \rightarrow \text{Im}\varphi \hookrightarrow V$ . Since both  $K$  and  $V$  are  $\mu$ -stable and have the same slope,  $\text{Im}\varphi$  must have the same slope as well. Furthermore,  $K \rightarrow \text{Im}\varphi$  must be an identity map, hence  $\text{Im}\varphi$  is locally free because  $K$  is locally free. Since  $V$  is also locally free,  $\text{Im}\varphi \hookrightarrow V$  must also be an identity map. This concludes  $K = V$ , which means the exact sequence (2) splits. Contradiction!

This shows that  $F$  must be  $\mu$ -semistable.  $\square$

Now we want to prove the extension in the above lemma is essentially unique for a generic choice of  $G$ .

**Lemma 3.8.** *there is an open dense subset  $U_2$  of the locally free  $\mu$ -stable locus in  $\mathfrak{M}_H(2v_2)$ , such that for any  $G \in U_2$ , we have  $\dim \text{Ext}^1(G, E) = 2$ . In other words, there is only one such sheaf  $F$  up to isomorphisms as in lemma 3.7.*

*Proof.* We have  $\chi(G, E) = -\langle v(G), v(E) \rangle = -\langle 2v_2, v_0 \rangle = -2$ . However  $\text{Hom}(G, E) = 0$  since  $\frac{\text{deg } G}{\text{rank } G} > \frac{\text{deg } E}{\text{rank } E}$ . By Serre duality, we have  $\text{Ext}^2(G, E) = \text{Hom}(E, G)$ . Therefore it suffices to show that for every  $G \in U_2$  we have  $\text{Hom}(E, G) = 0$ .

Since the moduli space  $\mathfrak{M}_H(2v_2)$  is irreducible [KLS06, Theorem 4.4], by the upper semi-continuity theorem, the vanishing of  $\text{Hom}$  is an open condition, therefore it suffices to explicitly produce one example of a semistable sheaf  $G' \in \mathfrak{M}_H(2v_2)$ , for which  $\text{Hom}(E, G') = 0$ . Note that  $G'$  doesn't have to be locally free or strictly stable. In fact, we will show that, for a generic choice of  $G_0 \in \mathfrak{M}_H(v_2)$ , we have  $\text{Hom}(E, G_0) = 0$ , therefore  $G' = G_0 \oplus G_0$  will solve the problem.

Since  $v_0, v_1$  and  $v_2$  are all primitive, we are now in a very favourable situation, which has been thoroughly studied in [Yos99b, Section 3]. In fact, in [Yos99b, Lemma 3.1], it is proved that sheaves  $F_0 \in \mathfrak{M}_H(v_1)$  satisfying  $\dim \text{Hom}(E, F_0) = 1$  form an open dense subset  $W_1$  of  $\mathfrak{M}_H(v_1)$ . Moreover, by the last paragraph in the proof of [Yos99b, Theorem 3.5],  $W_1$  is isomorphic to an open dense subset  $W_2$  of  $\mathfrak{M}_H(v_2)$ , consisting of all  $G_0 \in \mathfrak{M}_H(v_2)$  satisfying  $\dim \text{Hom}(E, G_0) = 0$ . Therefore for any such  $G_0$ , the self direct sum  $G' = G_0 \oplus G_0$  satisfies  $\dim \text{Hom}(E, G') = 0$ . As said above, the upper semi-continuity theorem guarantees the same is true on an open dense subset  $U_2$  of the locally free  $\mu$ -stable locus in  $\mathfrak{M}_H(2v_2)$ .  $\square$

Now we are ready to finish the proof of proposition 3.1.

*Proof of proposition 3.1.* For every locally free  $\mu$ -stable sheaf  $G \in U_2$ , by lemma 3.7 and 3.8, we have associated it a locally free  $\mu$ -semistable sheaf  $F$  with Mukai vector

$2v_1$ , which establishes a morphism from  $U_2$  to the moduli space  $\mathfrak{M}_{X,H}^{\mu\text{-ss}}(2v_1)$  of  $\mu$ -semistable sheaves with Mukai vector  $2v_1$ . In fact, the extension sequence in lemma 3.7 holds in flat families. Let  $V$  be any affine open subscheme of  $U_2$ . Let  $\mathcal{G}$  be the universal sheaf on  $V \times X$ , and let  $\mathcal{E}$  be the pullback of the exceptional bundle  $E$  along the projection from  $V \times X$  to  $X$ . Then we have the extension sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

on  $V \times X$ , which induces a morphism from  $V$  to  $\mathfrak{M}_{X,H}^{\mu\text{-ss}}(2v_1)$ . Furthermore, the restriction of the above extension sequence to every closed point in  $\mathcal{F}$  is exactly the extension sequence in lemma 3.7. Therefore, the morphism from all the affine subschemes of  $U_2$  glue together to give a morphism

$$\eta : U_2 \longrightarrow \mathfrak{M}_{X,H}^{\mu\text{-ss}}(2v_1).$$

Now we show this map is injective on closed points. From the exact sequence

$$0 \longrightarrow E^{\oplus 2} \longrightarrow F \longrightarrow G \longrightarrow 0,$$

we have

$$0 \longrightarrow \text{Hom}(E, E^{\oplus 2}) \longrightarrow \text{Hom}(E, F) \longrightarrow \text{Hom}(E, G).$$

In lemma 3.8 we have proved  $\text{Hom}(E, G) = 0$ , so  $\dim \text{Hom}(E, F) = \dim \text{Hom}(E, E^{\oplus 2}) = 2$ , which implies that there is only one way to get a quotient sheaf  $G$ .

It is easy to see that the locus  $U_1$  of  $\mu$ -stable locally free sheaves with Mukai vector  $2v_1$  is also an open subscheme of  $\mathfrak{M}_{X,H}^{\mu\text{-ss}}(2v_1)$ . Therefore  $\eta^{-1}(U_1) \cap U_2$  is an open subscheme of  $U_2$ , and hence an open subscheme of  $\mathfrak{M}_{X,H}(2v_2)$ . Since both  $U_1$  and  $U_2$  are smooth, the restriction of  $\eta$  on  $\eta^{-1}(U_1) \cap U_2$  is an isomorphism onto its image, which identifies two smooth open subschemes of  $\mathfrak{M}_{X,H}(2v_1)$  and  $\mathfrak{M}_{X,H}(2v_2)$ . By [KLS06, Theorem 4.4], the two moduli spaces are irreducible of the same dimension, so we can conclude that  $\mathfrak{M}_{X,H}(2v_1)$  and  $\mathfrak{M}_{X,H}(2v_2)$  are birational.  $\square$

Finally, we deal with the exceptional case  $r = 2$ . We have a similar birational result as in the above general case.

**Proposition 3.9.** *For any given  $s \geq 1$ , let  $X$  be a projective K3 surface with  $\text{Pic}(X) = \mathbb{Z}[H]$  where  $H$  is an ample line bundle with  $H^2 = 50s - 28$ . Consider Mukai vectors*

$$\begin{aligned} v_0 &= (5, 2H, 20s - 11), \\ v_1 &= (7, 3H, 32s - 18), \\ v_2 &= (2, H, 12s - 17). \end{aligned}$$

*Then the moduli spaces of semi-stable sheaves  $\mathfrak{M}_{X,H}(2v_1)$  is birational equivalent to  $\mathfrak{M}_{X,H}(2v_2)$ .*  $\square$

The proof of the above proposition is completely parallel to the proof of proposition 3.1. We will not repeat it here. Combining corollary 2.6, proposition 3.1 and 3.9, we have proved theorem 1.1.

## 4. RANK 0 CASE

In this section we will prove the following birational equivalence between a moduli space of rank 0 sheaves and a moduli space of rank 2 sheaves.

**Theorem 4.1.** *Let  $X$  be a K3 surface and  $H$  be an ample line bundle. Assume  $\text{Pic}(X) = \mathbb{Z}[H]$ . Let  $v_1 = (0, 2H, -2)$  and  $v_2 = (2, 2H, 0)$ . Then the moduli spaces of sheaves  $\mathfrak{M}_{X,H}(v_1)$  and  $\mathfrak{M}_{X,H}(v_2)$  are birational.*

*Proof.* Let  $U = \{i_*Q \in \mathfrak{M}_{X,H}(v_1) \mid Q \text{ is a line bundle supported on a smooth curve } C \in |2H|\}$ . Then  $U$  is an open subscheme of  $\mathfrak{M}_{X,H}(v_1)$ . We know that  $g(C) = 1 + \frac{1}{2}C^2 = 1 + 2H^2$ . Furthermore, by Grothendieck-Riemann-Roch formula, we know that

$$i_*(\text{ch}(Q) \cdot \text{td}(C)) = \text{ch}(i_*Q) \cdot \text{td}(X),$$

that is,

$$i_*((1, \deg Q) \cdot (1, -2H^2)) = (0, 2H, -2) \cdot (1, 0, 2).$$

We can conclude that  $\deg Q = 2H^2 - 2$ .

For every  $i_*Q \in U \subset \mathfrak{M}_{X,H}(v_1)$ , we construct a sheaf  $F$  as a non-trivial extension

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow F \longrightarrow i_*Q \longrightarrow 0.$$

First of all, we have to show that, for a generic  $i_*Q \in U$ , there is only one way to build up such an extension  $F$ . In order to prove this, we need to show that a generic  $i_*Q \in U$  satisfies  $\dim \text{Ext}^1(i_*Q, \mathcal{O}) = 2$ .

By Serre duality, we know that

$$\begin{aligned} \text{Ext}^1(i_*Q, \mathcal{O}) &= \text{Ext}^1(\mathcal{O}, i_*Q \otimes K_X)^\vee = H^1(X, i_*Q \otimes K_X)^\vee \\ &= H^1(C, Q \otimes i^*K_X)^\vee = H^0(C, Q^\vee \otimes N_{C/X}). \end{aligned}$$

We can find that  $\deg(Q^\vee \otimes N_{C/X}) = \deg(Q^\vee \otimes K_C) = 2H^2 + 2$ . By Riemann-Roch formula, we know that

$$\chi(Q^\vee \otimes N_{C/X}) = 1 - (2H^2 + 1) + (2H^2 + 2) = 2.$$

We need to prove that for a generic  $i_*Q \in U$ ,  $\dim H^0(C, Q^\vee \otimes N_{C/X}) = 2$ . By upper-semi-continuity theorem, we only need to show that there exists at least one  $i_*Q \in U$ , which makes  $\dim H^0(C, Q^\vee \otimes N_{C/X}) = 2$ .

Since  $Q$  is an arbitrary line bundle on an arbitrary smooth curve  $C \in |2H|$ , we only need to prove the following lemma:

**Lemma 4.2.** *Let  $C$  be a smooth curve and  $L$  be a line bundle on  $C$  with  $\chi(L) = 2$ . If  $L$  is general, then  $\dim H^0(L) = 2$  and  $\dim H^1(L) = 0$ .*

*Proof.* In contrary we assume the infimum of  $\dim H^0(L)$  of such  $L$ 's is at least 3. Pick any  $p \in C$  which is not a base point of  $L$ . From the exact sequence

$$0 \longrightarrow L(-p) \longrightarrow L \longrightarrow \mathcal{O}_p \longrightarrow 0$$

and its associated long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(L(-p)) \longrightarrow H^0(L) \longrightarrow \mathbb{C}_p \\ &\longrightarrow H^1(L(-p)) \longrightarrow H^1(L) \longrightarrow 0 \end{aligned}$$

we know that  $\dim H^0(L(-p)) = \dim H^0(L) - 1$ , and  $\dim H^1(L(-p)) = \dim H^1(L) \geq 1$ .

By Serre duality, we have  $H^1(L(-p)) = H^0(K_C \otimes L^\vee(p))^\vee$ . We can again pick a point  $q \in C$  which is not a base point of  $K_C \otimes L^\vee(p)$ . Then from the short exact sequence

$$0 \longrightarrow K_C \otimes L^\vee(p-q) \longrightarrow K_C \otimes L^\vee(p) \longrightarrow \mathcal{O}_q \longrightarrow 0$$

and its associated long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(K_C \otimes L^\vee(p-q)) \longrightarrow H^0(K_C \otimes L^\vee(p)) \longrightarrow \mathbb{C}_q \\ &\longrightarrow H^1(K_C \otimes L^\vee(p-q)) \longrightarrow H^1(K_C \otimes L^\vee(p)) \longrightarrow 0 \end{aligned}$$

we know that

$$\dim H^0(K_C \otimes L^\vee(p-q)) = \dim H^0(K_C \otimes L^\vee(p)) - 1$$

and

$$\dim H^1(K_C \otimes L^\vee(p-q)) = \dim H^1(K_C \otimes L^\vee(p)).$$

Again by Serre duality, we know that

$$\dim H^1(L(-p+q)) = \dim H^1(L(-p)) - 1 = \dim H^1(L) - 1$$

and

$$\dim H^0(L(-p+q)) = \dim H^0(L(-p)) = \dim H^0(L) - 1.$$

So we see that the existence of the line bundle  $L(-p+q)$  conflicts the assumption. Therefore, there must be a line bundle  $L$  with  $\chi(L) = 2$ , such that  $\dim H^0(L) = 2$ . By the upper-semi-continuity theorem, we know that this is true for a generic line bundle on any smooth curve  $C$ .  $\square$

Back to the proof of theorem 4.1. We know for a generic  $i_*Q \in U$ , we have  $\dim \text{Ext}^1(i_*Q, \mathcal{O}) = 2$ . Therefore, there is only one way to produce a sheaf  $F \in \mathfrak{M}_{X,H}(v_3)$  from  $i_*Q$  via the extension

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow F \longrightarrow i_*Q \longrightarrow 0.$$

Now we have to prove  $F$  is torsion free. In fact, we want to prove, if  $i_*Q \in U$  is generic, then  $F$  is locally free. We need the following two lemmas:

**Lemma 4.3.** *Under the above conditions, the sheaf  $F$  is locally free if and only if  $Q^\vee \otimes N_{C/X}$  is base point free.*

*Proof.* Note that  $\mathcal{H}om(i_*Q, \mathcal{O}^{\oplus 2}) = 0$ . By local to global spectral sequence, we have an isomorphism

$$\text{Ext}^1(i_*Q, \mathcal{O}^{\oplus 2}) \cong H^0(\mathcal{E}xt^1(i_*Q, \mathcal{O}^{\oplus 2})) = H^0(\mathcal{E}xt^1(i_*Q, \mathcal{O}) \oplus \mathcal{E}xt^1(i_*Q, \mathcal{O})).$$



Therefore, to prove the sheaf  $F$  is locally free, we only need to show the restriction of the two global sections of  $\mathcal{E}xt^1(i_*Q, \mathcal{O})$  to every local ring, provide a free local extension of the two sheaves.

Let  $p$  be a closed point of  $C$ . Over  $\text{Spec } \mathcal{O}_p$ , we can assume the curve is cut by one equation  $g \in \mathcal{O}_p$ , and resolve the torsion sheaf  $i_*Q$  as

$$0 \longrightarrow \mathcal{O}_p \xrightarrow{\cdot g} \mathcal{O}_p \longrightarrow \mathcal{O}_p/(g) \longrightarrow 0.$$

Apply the functor  $\text{Hom}(-, \mathcal{O}_p^{\oplus 2})$  we get the long exact sequence

$$0 \longrightarrow \text{Hom}(\mathcal{O}_p, \mathcal{O}_p^{\oplus 2}) \longrightarrow \text{Hom}(\mathcal{O}_p, \mathcal{O}_p^{\oplus 2}) \longrightarrow \text{Ext}^1(\mathcal{O}_p/(g), \mathcal{O}_p^{\oplus 2}) \longrightarrow 0.$$

Therefore, let  $(\varphi_1, \varphi_2)$  be the restriction of two linearly independent sections of  $\mathcal{E}xt^1(i_*Q, \mathcal{O})$  to the local ring  $\mathcal{O}_p$ , then  $(\varphi_1, \varphi_2) \in \mathcal{E}xt^1(i_*Q, \mathcal{O}^{\oplus 2})_p = \text{Ext}^1(\mathcal{O}_p/(g), \mathcal{O}_p^{\oplus 2})$ . They are images of functions  $(f_1, f_2) \in \text{Hom}(\mathcal{O}_p, \mathcal{O}_p^{\oplus 2})$ . Then the stalk of the sheaf  $F$  over  $\text{Spec } \mathcal{O}_p$  is given by the push out diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_p & \xrightarrow{(f_1, f_2)} & \mathcal{O}_p^{\oplus 2} & \longrightarrow & \mathcal{O}_p/(g) \longrightarrow 0 \\ & & \downarrow g & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_p & \longrightarrow & F_p & \longrightarrow & \mathcal{O}_p/(g) \longrightarrow 0 \end{array}$$

Therefore, the stalk  $F_p = \mathcal{O}_p^{\oplus 3}/(f_1, f_2, g)$ . Since  $g$  is the defining equation of the curve  $C$ ,  $g(p) = 0$ . So  $F_p$  is a free  $\mathcal{O}_p$  module if and only if  $f_1(p)$  and  $f_2(p)$  are not simultaneously 0.

Note that  $(\varphi_1, \varphi_2)$  is the restriction of  $(f_1, f_2)$  on the curve  $C$ . So  $F_p$  is free when  $\varphi_1(p)$  and  $\varphi_2(p)$  are not simultaneously 0. Therefore, as long as  $\varphi_1$  and  $\varphi_2$  do not have common zeroes along  $C$ , the sheaf  $F$  is locally free. That is to say, we want the sheaf  $\mathcal{E}xt^1(i_*Q, \mathcal{O})$  to be base point free for a generic  $i_*Q$ .

Furthermore, we can also observe from the above that the sheaf  $\mathcal{E}xt^1(i_*Q, \mathcal{O})$  is canonically isomorphic to  $i_*\mathcal{H}om(Q \otimes \mathcal{O}_C(-C), \mathcal{O}_C) = i_*(Q^\vee \otimes N_{C/X})$ . So we have  $H^0(X, \mathcal{E}xt^1(i_*Q, \mathcal{O})) = H^0(C, Q^\vee \otimes N_{C/X})$ . Therefore, the extension  $F$  is locally free if and only if  $Q^\vee \otimes N_{C/X}$  is base point free.  $\square$

We have already proved that for a generic  $i_*Q \in U$ ,  $Q^\vee \otimes N_{C/X}$  has two dimensional global sections. The next step is to prove that under this condition, a generic  $Q^\vee \otimes N_{C/X}$  is base point free.

**Lemma 4.4.** *Let  $C$  be a smooth curve and  $L$  be a line bundle on  $C$  with  $\chi(L) = 2$ . For a general  $L$ , it is base point free.*

*Proof.* Without loss of generality, we only need to consider all line bundles  $L$  with  $\dim H^0(L) = 2$  and  $\dim H^1(L) = 0$ . Note that base point freeness is an open condition in a flat family of line bundles with constant dimensional cohomology groups. Therefore we only need to show that there is at least one such line bundle  $L$  which is base point free.

First of all we can choose a line bundle  $M$  such that  $\dim H^0(M) = 1$  and  $\dim H^1(M) = 0$ . In fact, for any line bundle  $L$  with  $\dim H^0(L) = 2$  and  $\dim H^1(L) = 0$ , let  $p$  be a point not in the base locus of  $L$ , then  $M = L(-p)$  does the job. We denote the generator of  $H^0(M)$  by  $\sigma$ .

We claim that there must be a point  $q$  which makes  $M(q)$  base point free. In fact, from the exact sequence of cohomology groups, we know  $\dim H^0(M(q)) = 2$  and  $\dim H^1(M(q)) = 0$  for any  $q \in C$ . We assume the contrary that for every  $q \in C$ ,  $M(q)$  has a base point. Then the base point must be in the zero locus of  $\sigma$ . Since the zero set of the section  $\sigma$  is finite, we conclude that there must be a point  $r \in \sigma^{-1}(0)$ , which is the common base point of  $M(q)$ 's for all  $q \in C$ . In particular,  $r$  is the base point of  $M(r)$  which is absurd. This proves the lemma.  $\square$

Now we come back again to the proof of theorem 4.1, following the argument before the above two lemmas. By these lemmas, we see that for a generic  $i_*Q \in U$ , the extension sheaf  $F$  we constructed is a locally free sheaf.

Now we prove the sheaf  $F$  is stable.

Assume that  $F$  can be destabilized by a subsheaf  $L$ . Since  $F$  is locally free of rank 2, without loss of generality we can further assume  $\text{rank } L = 1$  and  $L$  is locally free. From  $\frac{\deg L}{\text{rank } L} \geq \frac{\deg F}{\text{rank } F}$ , we have  $c_1(L) \geq H$ . We assume that  $L = \mathcal{O}(H)$ , since otherwise  $\mathcal{O}(H)$  also destabilizes  $F$ .

Obviously  $\text{Hom}(L, \mathcal{O}^{\oplus 2}) = 0$  since  $\frac{\deg L}{\text{rank } L} > 0$ . We also have

$$\text{Hom}(L, i_*Q) = H^0(X, L^\vee \otimes i_*Q) = H^0(C, i^*L^\vee \otimes Q).$$

Since  $\deg(i^*L^\vee \otimes Q) = \deg L^\vee + \deg Q = -L \cdot C + \deg Q = -2H^2 + (2H^2 - 2) = -2$ , we know that  $\text{Hom}(L, i_*Q) = H^0(C, i^*L^\vee \otimes Q) = 0$ . Therefore  $\text{Hom}(L, F) = 0$ . Contradiction!

Finally, we prove for every such sheaf  $F$  that we obtained by extension

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow F \longrightarrow i_*Q \longrightarrow 0,$$

there is only one sheaf  $i_*Q \in U$  with  $\dim H^0(C, Q^\vee \otimes N_{C/X}) = 2$  which produces  $F$ . In fact, in the long exact sequence of cohomology groups

$$0 \longrightarrow H^0(X, \mathcal{O}^{\oplus 2}) \longrightarrow H^0(X, F) \longrightarrow H^0(X, i_*Q),$$

we already know that

$$H^0(X, i_*Q) = H^0(C, Q) = H^1(C, Q^\vee \otimes K_C)^\vee = H^1(C, Q^\vee \otimes N_{C/X})^\vee = 0.$$

So  $\dim H^0(F) = \dim H^0(\mathcal{O}^{\oplus 2}) = 2$ . Therefore  $i_*Q$  is also uniquely determined by  $F$ .

In the same way as in the proof of proposition 3.1, we can see that the above extension can also be done in flat families, therefore gives a morphism

$$\eta : V \longrightarrow \mathfrak{M}_{X,H}(v_2)$$

where  $V$  is the open subscheme of  $\mathfrak{M}_{X,H}(v_1)$ , which parametrizes all line bundles supported on smooth curves in the linear system  $|2H|$  satisfying the conditions of

lemmas 4.2 and 4.4. And the image of  $\eta$  lies in the smooth open subscheme of  $\mathfrak{M}_{X,H}(v_2)$  which parametrizes locally free stable sheaves with Mukai vector  $v_2$ . The above discussion also shows that  $\eta$  identifies  $V$  with its image. Since the moduli spaces  $\mathfrak{M}_{X,H}(v_1)$  and  $\mathfrak{M}_{X,H}(v_2)$  are both irreducible and of the same dimension by [KLS06, Theorem 4.4],  $\eta$  establishes a birational map from  $\mathfrak{M}_{X,H}(v_1)$  to  $\mathfrak{M}_{X,H}(v_2)$ .  $\square$

## REFERENCES

- [Gie77] David Gieseker. On the moduli of vector bundles on an algebraic surface. *Ann. of Math.* (2), 106(1):45–60, 1977. [1](#)
- [HL97] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997. [2](#), [4](#), [5](#), [7](#)
- [Huy03] Daniel Huybrechts. The Kähler cone of a compact hyperkähler manifold. *Math. Ann.*, 326(3):499–513, 2003. [1](#), [2](#)
- [KLS06] Dmitry Kaledin, Manfred Lehn, and Christoph Sorger. Singular symplectic moduli spaces. *Invent. Math.*, 164(3):591–614, 2006. [13](#), [14](#), [19](#)
- [Mar77] Masaki Maruyama. Moduli of stable sheaves. I. *J. Math. Kyoto Univ.*, 17(1):91–126, 1977. [1](#)
- [Mar78] Masaki Maruyama. Moduli of stable sheaves. II. *J. Math. Kyoto Univ.*, 18(3):557–614, 1978. [1](#)
- [Muk84] Shigeru Mukai. Symplectic structure of the moduli space of sheaves on an abelian or  $K3$  surface. *Invent. Math.*, 77(1):101–116, 1984. [1](#)
- [O’G99] Kieran O’Grady. Desingularized moduli spaces of sheaves on a  $K3$ . *J. Reine Angew. Math.*, 512:49–117, 1999. [2](#)
- [Per11] Arvid Perego. A note on deformations of moduli spaces of sheaves on  $K3$  surfaces. *ArXiv preprint*, February 2011. ArXiv:1102.1847 [math.AG]. [3](#)
- [PR10] Arvid Perego and Antonio Rapagnetta. Deformation of the O’Grady moduli spaces. *ArXiv preprint*, August 2010. ArXiv:1008.0190 [math.AG]. [3](#)
- [Yos99a] Kōta Yoshioka. Irreducibility of moduli spaces of vector bundles on  $k3$  surfaces. *Arxiv preprint*, July 1999. ArXiv:math/9907001v2. [10](#)
- [Yos99b] Kōta Yoshioka. Some examples of Mukai’s reflections on  $K3$  surfaces. *J. Reine Angew. Math.*, 515:97–123, 1999. [2](#), [9](#), [10](#), [13](#)
- [Yos01a] Kōta Yoshioka. A note on Fourier-Mukai transform. *ArXiv preprint*, December 2001. ArXiv:math/0112267. [3](#)
- [Yos01b] Kōta Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Math. Ann.*, 321(4):817–884, 2001. [2](#), [5](#), [10](#)

MAX PLANCK INSTITUTE FOR MATHEMATICS, VIVATSGASSE 7, 53111 BONN, GERMANY

*Current address:* Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, United Kingdom

*E-mail address:* [zz505@bath.ac.uk](mailto:zz505@bath.ac.uk)