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Stability of some vector bundles on Hilbert schemes of points on K3 surfaces

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Abstract Let X be a projective K3 surfaces. In two examples where there exists a fine moduli space M of stable vector bundles on X, isomorphic to a Hilbert scheme of points, we prove that the universal family $\mathcal{E}$ on $X \times M$ can be understood as a complete flat family of stable vector bundles on M parametrized by X, which identifies X with a smooth connected component of some moduli space of stable sheaves on M.

Keywords stable sheaves, moduli spaces, universal families, Hilbert schemes

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Introduction

Let X be a projective K3 surface, and M a moduli space of semistable sheaves on X. By Mukai's seminal work [15], when M is smooth, it is an example of the so-called irreducible holomorphic symplectic manifolds, which are an important class of building blocks in the classification of compact Kähler manifolds with trivial first Chern class. It is then an interesting question to understand whether the moduli spaces $\mathcal{M}$ of semistable sheaves on M inherit any good properties from M. This paper grew out of an attempt to study this question. When $\dim M > 2$, we cannot expect $\mathcal{M}$ to carry a holomorphic symplectic structure in general, because the Serre duality does not induce a non-degenerate anti-symmetric pairing on the tangent space of $\mathcal{M}$ any more, as opposed to the case of K3 surfaces; however, some components of $\mathcal{M}$ may nevertheless be holomorphic symplectic.

In order to study this question, we need to classify all semistable sheaves on M with fixed Chern classes, which seems difficult in general when $\dim M > 2$; it is even a challenging question to construct any non-trivial examples of semistable sheaves on M, due to the fact that stability is difficult to check on higher dimensional varieties in general. When M is a Hilbert scheme of points on the K3 surface X, a natural family of vector bundles on M for considering stability are the so-called tautological bundles, which were proven to be stable with respect to a suitable choice of an ample line bundle on M by Schlickewei [18], Wandel [21] and Stapleton [20]. In fact, Wandel proved that, under some mild assumptions, the connected component of the moduli space containing the tautological bundles is isomorphic to some moduli space of vector bundles on the underlying K3 surface X.

There is another way to construct examples of stable sheaves on M. Assuming that M is a fine moduli space of stable sheaves on X with a universal family $\mathcal{E}$ on $X \times M$, and denoting the "wrong-way fiber" $\mathcal{E}|_{\{x\}\times M}$ by E_x for each closed point $x \in X$, we can ask the following questions:

- Is $\mathcal{E}$ also a flat family of coherent sheaves on M parametrized by X?
- If so, are the "wrong-way" fibers E_x stable sheaves on M with respect to some suitable choice of an ample line bundle for every closed point $x \in X$?

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- If so, can we identify X with a connected component of the corresponding moduli space of stable sheaves on M?

This idea has also been explored in the literature. In [17], the authors studied some families of ideal sheaves and torsion sheaves of pure dimension 1, and obtained an affirmative answer to the above questions in these cases. A systematic study of the above questions in the case of locally free sheaves was carried out in the very interesting and inspiring thesis of Wray [22]. In order to get around the difficulty of proving stability directly, he invoked the very deep and powerful technique of Hitchin-Kobayashi correspondence to translate the stability problem to the existence of some Hermitian-Einstein metrics, which was then solved by analytic methods to give affirmative answers to the above questions.

The present paper is devoted to study the above questions, in particular the stability of wrong-way fibers E_x with respect to a polarization near the boundary of the ample cone of M, in the very classical way by showing that every proper subsheaf of E_x of a smaller rank has a smaller slope. We will focus on two special cases, namely a projective K3 surface X along with a Mukai vector v such that either

- $\mathrm{NS}(X) = \mathbb{Z}h$ with $h^2 = 4k$ and $v = (k+1, -h, 1)$ for any $k \geqslant 1$; or
- $\mathrm{NS}(X) = \mathbb{Z}e \oplus \mathbb{Z}f$ with the intersection matrix given by $\begin{pmatrix} -2k & 2k+1 \\ 2k+1 & 0 \end{pmatrix}$ for any $k \geqslant 2$ as well as $v = (2k-1, e+(2k-1)f, 2k)$.

We summarize our main results in the following theorem:

Theorem 0.1 *For any projective K3 surface X satisfying either of the above conditions,*

(1) we can explicitly construct a fine moduli space M of stable vector bundles of Mukai vector v on X, isomorphic to the Hilbert scheme of k points on X, along with a universal family $\mathcal{E}$ (see Theorem 2.3 and Theorem 3.7);
(2) there exists an ample divisor H on M such that $\mathcal{E}$ can be regarded as a flat family of μ_H-stable vector bundles on M parametrized by X (see Theorem 2.8 and Theorem 3.15);
(3) the classifying morphism induced by the family $\mathcal{E}$ identifies X with a smooth connected component of a moduli space of μ_H-stable sheaves on M (see Theorem 2.10 and Theorem 3.16).

Let us briefly explain how we achieved the above results. Our choices of the K3 surfaces and the Mukai vectors, as well as the explicit constructions of the moduli space M and the universal family $\mathcal{E}$ in the above two cases, are motivated by [10, Example 5.3.7] and [16, Theorem 1.2] respectively. In fact, in both cases, the stable sheaves on X are given by the spherical twist (or its inverse) of the ideal sheaves of k points on X around $\mathcal{O}_X$, hence their corresponding moduli spaces M are isomorphic to the Hilbert scheme $X^{[k]}$ of k points on X. To show the slope stability of the wrong-way fibers E_x with respect to some ample divisor H on M, we apply the technique developed by Stapleton [20]; namely, we first prove the slope stability of E_x with respect to a natural nef divisor on M by passing to the k-fold product of X, then use the openness of stability to perturb the nef divisor to a nearby ample divisor. In fact, since the perturbation argument in [20] works only for individual sheaves, we need to generalize it so as to find an ample divisor H with respect to which all E_x's are simultaneously stable. Finally, to identify X as a smooth connected component of some moduli space of stable sheaves on M, we interpret E_x's as images of some sheaves or derived objects on X under the integral functor Φ induced by the universal ideal sheaf for $X^{[k]}$. By the fundamental result of Addington [1] that Φ is a $\mathbb{P}^{k-1}$-functor, we can obtain, by computing the relevant cohomology groups, that E_x's are distinct and the tangent space of deformations of each E_x is of dimension 2, which leads immediately to the conclusion.

The text is organized in three sections. The first section gives background on integral functors, while the other two deal with the two cases mentioned above respectively. All objects in this text are defined over the field of complex numbers $\mathbb{C}$.

1 Background on spherical twists and $\mathbb{P}^n$-functors

Let X denote a smooth projective variety with $\dim(X) = d$. As we will need them later, we quickly recall some facts about spherical twists and $\mathbb{P}^n$-functors in this section.

Definition 1.1 *An object $\mathcal{S} \in \mathrm{D^b}(X)$ is called spherical if*

i) $\mathcal{S} \otimes \omega_X \cong \mathcal{S}$

ii) $\mathrm{Ext}^i(\mathcal{S}, \mathcal{S}) = \begin{cases} \mathbb{C} & \textit{if } i = 0, d \\ 0 & \textit{otherwise} \end{cases}$

Remark 1.2 We note the fact that if X is a K3 surface, then any $L \in \mathrm{Pic}(X)$ is spherical.

Using spherical objects one can construct autoequivalences of $\mathrm{D^b}(X)$ in the following way: to any object $\mathcal{F} \in \mathrm{D^b}(X)$ one can associate the following object in $\mathrm{D^b}(X \times X)$:

$$\mathcal{P}_{\mathcal{F}} := \mathrm{Cone}(\mathcal{F}^\vee \boxtimes \mathcal{F} \longrightarrow \mathcal{O}_\Delta).$$

We refer to [8, §8] for an exact description of the map $\mathcal{F}^\vee \boxtimes \mathcal{F} \to \mathcal{O}_\Delta$ and more information.

Definition 1.3 *The spherical twist associated to a spherical object $\mathcal{S} \in \mathrm{D^b}(X)$ is the Fourier-Mukai transform*

$$T_{\mathcal{S}} := \Phi_{\mathcal{P}_{\mathcal{S}}} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X)$$

with kernel $\mathcal{P}_{\mathcal{S}}$.

The most important fact about the spherical twist is

Proposition 1.4 *Let $\mathcal{S}$ be a spherical object in $\mathrm{D^b}(X)$. Then the induced spherical twist*

$$T_{\mathcal{S}} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X)$$

is an autoequivalence.

The first proof of this proposition was given by Seidel and Thomas, see [19, Theorem 1.2].

Remark 1.5 By [8, Exercise 8.5] the effect of the spherical twist $T_{\mathcal{S}}$ on an object $\mathcal{G} \in \mathrm{D^b}(X)$ can be described by the following distinguished triangle:

$$T_{\mathcal{S}}(\mathcal{G})[-1] \longrightarrow R\,\mathrm{Hom}(\mathcal{S},\mathcal{G}) \otimes \mathcal{S} \longrightarrow \mathcal{G} \longrightarrow T_{\mathcal{S}}(\mathcal{G}).$$

As the spherical twist $T_{\mathcal{S}}$ is an autoequivalence one can also study the inverse $T_{\mathcal{S}}^{-1}$. For any object $\mathcal{G} \in \mathrm{D^b}(X)$ there exists the following distinguished triangle, see [8, Remark 8.11]:

$$T_{\mathcal{S}}^{-1}(\mathcal{G}) \longrightarrow \mathcal{G} \longrightarrow R\,\mathrm{Hom}(\mathcal{S},\mathcal{G}) \otimes \mathcal{S}[d] \longrightarrow T_{\mathcal{S}}^{-1}(\mathcal{G})[1].$$

We are also interested in another class of integral functors, the so-called $\mathbb{P}^n$-functors, which were introduced by Addington in a very general setting in [1, §4]. We will only need the following special example:

Example 1.6 Let X be a K3 surface, then the integral functor

$$\Phi\colon\ \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X^{[k]})$$

whose kernel is the universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ on $X \times X^{[k]}$ is a $\mathbb{P}^{k-1}$-functor with corresponding autoequivalence $H = [-2]$ by [1, Theorem 3.1, Example 4.2(2)].

Remark 1.7 The fact that the above integral functor Φ is a $\mathbb{P}^{k-1}$-functor with the corresponding autoequivalence $H = [-2]$ has the following useful consequence, see [2, §2.1]: for any $E, F \in \mathrm{D^b}(X)$ we have an isomorphism of graded vector spaces

$$\mathrm{Ext}^*_{X^{[k]}}(\Phi(E), \Phi(F)) \cong \mathrm{Ext}^*_X(E,F) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{C}).$$

2 K3 surfaces with Picard number one

Throughout this section we assume X is a K3 surface such that $\mathrm{NS}(X) = \mathbb{Z}h$, where h is an ample class with $h^2 = 4k$. We denote the line bundle associated to h by $\mathcal{O}_X(1)$ and the Hilbert scheme of length k subschemes of X by $X^{[k]}$.

2.1 Explicit construction of a universal family

In this subsection we generalize [10, Example 5.3.7] to give an explicit construction of a universal family of stable vector bundles on X parametrized by the Hilbert scheme $X^{[k]}$ for $k \geqslant 1$. Let h be the ample generator of $\mathrm{NS}(X)$ and $v = (k+1, -h, 1) \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$. We have the following facts:

Lemma 2.1 *The moduli space $M_h(v)$ of μ_h-stable sheaves on X with Mukai vector v is a smooth projective variety of dimension $2k$ and a fine moduli space. Furthermore every point $[E] \in M_h(v)$ represents a locally free sheaf.*

Proof. We note that every μ_h-semistable sheaf E with $v(E) = v$ is μ_h-stable as $\rho(X) = 1$. Thus $M_h(v)$ is a smooth projective variety. We compute:

$$\dim(M_h(v)) = v^2 + 2 = 4k - 2(k+1) + 2 = 2k.$$

Furthermore $v' = (k+1, -h, a)$ with $a \geqslant 2$ satisfies

$$v'^2 + 2 = 4k - 2a(k+1) + 2 \leqslant 4k - 4(k+1) + 2 = -2 < 0,$$

and thus the second Chern class is minimal (here $c_2(E) = 3k$). This minimality implies that every point $[E]$ in $M_h(v)$ is given by a locally free sheaf E. The condition $\gcd(k+1, 1) = 1$ implies that $M_h(v)$ is a fine moduli space by [10, Remark 4.6.8]. □

The following lemma produces examples of elements in this moduli space:

Lemma 2.2 *For any $[Z] \in X^{[k]}$ the sheaf $I_Z(1)$ is globally generated, i.e. the evaluation morphism*

$$\mathrm{ev} : H^0(I_Z(1)) \otimes \mathcal{O}_X \to I_Z(1)$$

is surjective. Furthermore $E_Z := \ker(\mathrm{ev})$ is a μ_h-stable locally free sheaf with Mukai vector given by $v(E_Z) = (k+1, -h, 1)$.

Proof. The standard exact sequence

$$0 \longrightarrow I_Z(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_Z(1) \longrightarrow 0 \tag{1}$$

shows

$$\chi(I_Z(1)) = \chi(\mathcal{O}_X(1)) - \chi(\mathcal{O}_Z(1)) = (2k+2) - k = k+2.$$

Since Z has codimension two in X, using Serre duality gives

$$H^2(I_Z(1)) \cong \mathrm{Hom}(I_Z(1), \mathcal{O}_X)^\vee \cong H^0(\mathcal{O}_X(-1))^\vee = 0.$$

By [5, Proposition 3.7], the line bundle $\mathcal{O}_X(1)$ is k-very ample which implies that the exact sequence of global sections attached to (1)

$$0 \longrightarrow H^0(I_Z(1)) \longrightarrow H^0(\mathcal{O}_X(1)) \longrightarrow H^0(\mathcal{O}_Z(1)) \longrightarrow 0$$

is still exact. This implies $H^1(I_Z(1)) \cong H^1(\mathcal{O}_X(1)) = 0$ and thus

$$\dim(H^0(I_Z(1))) = \chi(I_Z(1)) = k+2.$$

Now if the evaluation map is not surjective, let $Q := \mathrm{coker}(\mathrm{ev})$ and pick $x \in \mathrm{supp}(Q)$. Then we have an exact sequence

$$0 \longrightarrow I_{Z'}(1) \longrightarrow I_Z(1) \longrightarrow \mathcal{O}_x \longrightarrow 0$$

for a length $k+1$ subscheme Z' containing Z.

Since $I_Z(1)$ is not globally generated at x the last exact sequence gives isomorphisms

$$H^0(I_{Z'}(1)) \cong H^0(I_Z(1)) \text{ and } H^1(I_{Z'}(1)) \cong H^0(\mathcal{O}_x) \neq 0.$$

But $\mathcal{O}_X(1)$ is k-very ample so by definition

$$0 \longrightarrow H^0(I_{Z'}(1)) \longrightarrow H^0(\mathcal{O}_X(1)) \longrightarrow H^0(\mathcal{O}_{Z'}(1)) \longrightarrow 0$$

is still exact, which implies $H^1(I_{Z'}(1)) = 0$, a contradiction. So ev is indeed surjective and we have an exact sequence:

$$0 \longrightarrow E_Z \longrightarrow H^0(I_Z(1)) \otimes \mathcal{O}_X \longrightarrow I_Z(1) \longrightarrow 0. \tag{2}$$

Computing invariants shows $\mathrm{rk}(E_Z) = k+1$, $c_1(E_Z) = -h$ and $c_2(E_Z) = 3k$, hence indeed $v(E_Z) = (k+1, -h, 1)$. The sheaf E_Z is locally free as it is the kernel of a morphism between a locally free and a torsion free sheaf on a smooth surface. The stability of E_Z follows from [23, Lemma 2.1 (2-2)]. □

We can globalize the construction in Lemma 2.2: let $\mathcal{Z} \subset X \times X^{[k]}$ denote the universal length k subscheme, $\mathcal{I}_{\mathcal{Z}}$ its ideal sheaf. There are projections $p : X \times X^{[k]} \to X^{[k]}$ as well as $q : X \times X^{[k]} \to X$. Define a sheaf $\mathcal{E}$ on $X \times X^{[k]}$ by the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow p^*(p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1))) \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1) \longrightarrow 0. \tag{3}$$

Then $\mathcal{E}$ is p-flat and $\mathcal{E}_{|p^{-1}(Z)} \cong E_Z$, which implies that $\mathcal{E}$ is locally free on $X \times X^{[k]}$ by [10, Lemma 2.1.7]. Thus $\mathcal{E}$ defines a classifying morphism

$$\varphi : X^{[k]} \to M_h(v),\ [Z] \mapsto [E_Z]\,.$$

In fact we have:

Theorem 2.3 *The classifying morphism $\varphi : X^{[k]} \to M_h(v)$ is an isomorphism.*

Proof. Looking at Remark 1.5 we see that the sheaf E_Z defined by the exact seqeunce (2) is nothing but the shifted spherical twist of $I_Z(1)$ around $\mathcal{O}_X$, more exactly we have

$$E_Z = T_{\mathcal{O}_X}(I_Z(1))[1],$$

similar to [9, Example 10.3.6]. By Proposition 1.4 the spherical twist $T_{\mathcal{O}_X}$ is an autoequivalence of $\mathrm{D^b}(X)$ likewise is the shift [1]. But then the classifying morphism

$$\varphi : X^{[k]} \to M_h(v),\ [Z] \mapsto [E_Z] = [T_{\mathcal{O}_X}(I_Z(1))[1]]$$

is a composition of autoequivalences and thus maps non-isomorphic objects to non-isomorphic objects, hence φ is injective on closed points. Since both $X^{[k]}$ and $M_h(v)$ are smooth of dimension $2k$ the morphism φ is an open embedding and thus an isomorphism as both spaces are irreducible. □

2.2 Stability of wrong-way fibers

In the above section, we explicitly constructed a universal family $\mathcal{E}$, which is a locally free sheaf on $X \times X^{[k]}$. In this section we take the alternative point of view and consider $\mathcal{E}$ as a family of vector bundles on $X^{[k]}$ parametrized by X. A "wrong-way fiber" of $\mathcal{E}$ is just the restriction of $\mathcal{E}$ over a point $x \in X$ which gives a locally free sheaf on $X^{[k]}$.

More precisely, we first note that by standard cohomology and base change arguments

$$p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \otimes \mathcal{O}_{[Z]} \to H^0(I_Z(1))$$

is an isomorphism. Hence

$$K := p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \tag{4}$$

is a locally free sheaf of rank $k+2$ on $X^{[k]}$. This implies that $\mathcal{E}$ is not only p-flat, but also q-flat since $\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)$ is both p- and q-flat by [14, Theorem 2.1]. Thus we can restrict the exact sequence (3) to the fiber over a point $x \in X$ and get the following description of the fiber $E_x := \mathcal{E}_{|q^{-1}(x)}$:

$$0 \longrightarrow E_x \longrightarrow K \longrightarrow I_{S_x} \longrightarrow 0, \tag{5}$$

where $S_x := \left\{[Z] \in X^{[k]} \mid x \in \mathrm{supp}(Z)\right\}$ is a codimension 2 subscheme of $X^{[k]}$. Hence E_x is a locally free sheaf of rank $k+1$ on $X^{[k]}$.

Before proving the stability of E_x with respect to some ample class $H \in \mathrm{NS}(X^{[k]})$, we recall that for any coherent sheaf F on X there is the associated coherent *tautological sheaf* $F^{[k]}$ on $X^{[k]}$ defined by

$$F^{[k]} := p_*\left(q^*F \otimes \mathcal{O}_{\mathcal{Z}}\right). \tag{6}$$

If F is locally free of rank r then $F^{[k]}$ is locally free of rank kr.

Also recall the well-known fact that $\mathrm{NS}(X^{[k]}) = \mathrm{NS}(X)_k \oplus \mathbb{Z}\delta$. Here d_k is the divisor class on $X^{[k]}$ induced by the divisor class d on X and δ is a divisor class on $X^{[k]}$ such that $2\delta = [E]$ where E is the exceptional divisor of the Hilbert-Chow morphism $X^{[k]} \to X^{(k)}$. In our case this reads

$$\mathrm{NS}(X^{[k]}) = \mathbb{Z}h_k \oplus \mathbb{Z}\delta.$$

Lemma 2.4 *We have $c_1(E_x) = -h_k + \delta$.*

Proof. There is the exact sequence:

$$0 \longrightarrow p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \longrightarrow p_* q^* \mathcal{O}_X(1) \longrightarrow p_*(\mathcal{O}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \longrightarrow 0$$

as $R^1 p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) = 0$ since $H^1(I_Z(1)) = 0$ for all $[Z] \in X^{[k]}$.

We also have

$$p_* q^* \mathcal{O}_X(1) \cong H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}}$$

and the sheaf $p_*(\mathcal{O}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1))$ is nothing but the tautological sheaf $\mathcal{O}_X(1)^{[k]}$ associated to $\mathcal{O}_X(1)$ on $X^{[k]}$. By [11, Remark 3.20.] we also have $H^0(\mathcal{O}_X(1)^{[k]}) = H^0(\mathcal{O}_X(1))$. Thus, the above exact sequence can be rewritten as

$$0 \longrightarrow K \longrightarrow H^0(\mathcal{O}_X(1)^{[k]}) \otimes \mathcal{O}_{X^{[k]}} \longrightarrow \mathcal{O}_X(1)^{[k]} \longrightarrow 0. \tag{7}$$

Using [21, Lemma 1.5] we get

$$c_1(K) = -c_1(\mathcal{O}_X(1)^{[k]}) = -h_k + \delta.$$

Now exact sequence (5) gives $c_1(E_x) = c_1(K) = -h_k + \delta$. □

To compute slopes on $X^{[k]}$ we need the following intersection numbers, which can, for example, be found in [21, Lemma 1.10]:

Lemma 2.5 *For the classes h_k and δ from $\mathrm{NS}(X^{[k]})$ we have:*

- $h_k^{2k} = \frac{(2k-1)!}{(k-1)!2^{k-1}}(h^2)^k = \frac{(2k-1)!2^{k+1}}{(k-1)!}k^k > 0$
- $h_k^{2k-1}\delta = 0.$

We also recall the notations introduced in [20, §1]. The ample divisor h on X naturally induces an ample divisor

$$h_{X^k} = \bigoplus_{i=1}^{k} q_i^* h$$

on X^k, where q_i denotes the projection from X^k to the i-th factor, as well as a semi-ample divisor h_k on $X^{[k]}$.

Moreover, we write $X_\circ^k$, $S^k X_\circ$ and $X_\circ^{[k]}$ for the loci of the relevant spaces parametrizing distinct points. Then the natural map

$$\overline{\sigma}_\circ : X_\circ^k \to X_\circ^{[k]}$$

is an étale cover and $j : X_\circ^k \to X^k$ is an open embedding. For any coherent sheaf F on $X^{[k]}$, we denote by $F_\circ$ the restriction of F to $X_\circ^{[k]}$, and define

$$(F)_{X^k} = j_*(\overline{\sigma}_\circ^*(F_\circ))$$

which is a torsion free coherent sheaf if F is.

Proposition 2.6 *The vector bundle K defined in (4) is slope stable with respect to h_k.*

Proof. We follow the idea in the proof of [20, Theorem 1.4].

Since $(-)_\circ$ and $\overline{\sigma}_\circ^*(-)$ are exact, and $j_*(-)$ is left exact, by applying these functors to (7) we obtain an exact sequence of $\mathfrak{S}_n$-invariant reflexive sheaves on X^k as follows

$$0 \longrightarrow (K)_{X^k} \longrightarrow (H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}})_{X^k} \xrightarrow{\varphi} (\mathcal{O}_X(1)^{[k]})_{X^k}$$

where φ is not necessarily surjective. It is clear that

$$(H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}})_{X^k} = H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k},$$

and we also have

$$(\mathcal{O}_X(1)^{[k]})_{X^k} = \bigoplus_{i=1}^k q_i^* \mathcal{O}_X(1)$$

by [20, Lemma 1.1]. Hence the above sequence becomes

$$0 \longrightarrow (K)_{X^k} \longrightarrow H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k} \xrightarrow{\varphi} \bigoplus_{i=1}^k q_i^* \mathcal{O}_X(1) \tag{8}$$

where φ is the evaluation map on $X_\circ^k$.

More precisely, for any set of closed points $(x_1, \dots, x_n) \in X^k$ with $x_i \neq x_j$, the morphism of fibers can be identified as

$$\begin{aligned} \varphi_{(x_1,\dots,x_k)} : H^0(\mathcal{O}_X(1)) &\longrightarrow \bigoplus_{i=1}^k \mathcal{O}_X(1)_{x_i} \\ s &\longmapsto (s(x_1), \dots, s(x_k)) \end{aligned}$$

Since for any non-trivial $s \in H^0(\mathcal{O}_X(1))$, there are always (many choices of) distinct points $(x_1, \dots x_k) \in X^k$ such that $(s(x_1), \dots, s(x_k)) \neq (0, \dots, 0)$, we conclude that the map of global sections

$$H^0(\varphi) : H^0(\mathcal{O}_X(1)) \longrightarrow H^0(\bigoplus_{i=1}^k q_i^* \mathcal{O}_X(1))$$

is injective. It follows by exact sequence (8) that $(K)_{X^k}$ has no global sections, that is

$$H^0((K)_{X^k}) = 0. \tag{9}$$

Note that φ is surjective on $X_\circ^k$, hence $\mathrm{coker}(\varphi)$ is supported on the big diagonal of X^k which is of codimension 2. It follows that

$$c_1((K)_{X^k}) = -\sum_{i=1}^k q_i^* h.$$

We claim that $(K)_{X^k}$ has no $\mathfrak{S}_k$-invariant subsheaf which is destabilizing with respect to h_{X^k}. Indeed, assume F is an $\mathfrak{S}_k$-invariant subsheaf of $(K)_{X^k}$, then for some $a \in \mathbb{Z}$:

$$c_1(F) = a(\sum_{i=1}^k q_i^* h).$$

If $a \leqslant -1$, then

$$c_1(F) h_{X^k}^{2k-1} \leqslant c_1((K)_{X^k}) h_{X^k}^{2k-1} < 0$$

Since $1 \leqslant \mathrm{rk}(F) < \mathrm{rk}((K)_{X^k})$, it follows that $\mu_{h_{X^k}}(F) < \mu_{h_{X^k}}((K)_{X^k})$, hence F is not destabilizing.

If $a = 0$, we choose a (not necessarily $\mathfrak{S}_k$-invariant) non-zero stable subsheaf $F' \subseteq F$ which has maximal slope with respect to h_{X^k} (e.g. one can take a stable factor in the first Harder-Narasimhan factor of F). Without loss of generality, we can assume F and F' are both reflexive. Since F' is also a subsheaf of $H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k}$, there must be a projection from $H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k}$ to a certain direct summand of it, such that the composition of the embedding and projection $F' \to H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k} \to \mathcal{O}_{X^k}$ is non-zero. Since $\mu_{X^k}(F') \geqslant \mu_{X^k}(F) = 0 = \mu_{X^k}(\mathcal{O}_{X^k})$, and $\mathcal{O}_{X^k}$ is also stable with respect to h_{X^k}, the map $F' \to \mathcal{O}_{X^k}$ must be injective, and its cokernel is supported on a locus of codimension at least 2. Since both are reflexive, we must have $F' = \mathcal{O}_{X^k}$. Therefore F, and consequently $(K)_{X^k}$, have non-trivial global sections. This contradicts (9).

If $a \geqslant 1$, F would be a subsheaf of the trivial bundle $H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k}$ of positive slope. Contradiction.

Finally, assume G is a reflexive subsheaf of K. Then $(G)_{X^k}$ is an $\mathfrak{S}_k$-invariant reflexive subsheaf of $(K)_{X^k}$. By the above claim we have $\mu_{h_{X^k}}((G)_{X^k}) < \mu_{h_{X^k}}((K)_{X^k})$. It follows by [20, Lemma 1.2] that $\mu_{h_k}(G) < \mu_{h_k}(K)$. Therefore K is slope stable with respect to h_k, as desired. □

Proposition 2.7 *For any closed point $x \in X$, the bundle E_x is slope stable with respect to h_k.*

Proof. By Lemma 2.4, we have $c_1(E_x) = c_1(K) = -h_k + \delta$. Therefore by Lemma 2.5

$$c_1(E_x)h_k^{2k-1} = c_1(K)h_k^{2k-1} = (-h_k + \delta)h_k^{2k-1} = -h_k^{2k} < 0.$$

Assume F is a destabilizing subsheaf of E_x with $1 \leqslant \mathrm{rk}(F) \leqslant k$ and $c_1(F) = ah_k + b\delta$ for some $a, b \in \mathbb{Z}$. Then

$$c_1(F)h_k^{2k-1} = ah_k^{2k}.$$

By the assumption and Proposition 2.6, we have the inequality

$$\mu_{h_k}(E_x) \leqslant \mu_{h_k}(F) < \mu_{h_k}(K),$$

which can be written as

$$\frac{-h_k^{2k}}{k+1} \leqslant \frac{ah_k^{2k}}{\mathrm{rk}(F)} < \frac{-h_k^{2k}}{k+2} \Longleftrightarrow -\frac{\mathrm{rk}(F)}{k+1} \leqslant a < -\frac{\mathrm{rk}(F)}{k+2} \text{ as } h_k^{2k} > 0.$$

Such an integer a cannot exist. Contradiction. Hence E_x is stable with respect to h_k. □

2.3 A smooth connected component

In this section, we will interpret the universal sheaf $\mathcal{E}$ defined in (3) as a family of stable sheaves on $X^{[k]}$ whose base is a smooth connected component of the corresponding moduli space. We have shown above that each wrong-way fiber E_x of the family $\mathcal{E}$ is μ_{h_k}-stable; however, it would be more preferable to establish the stability with respect to some ample class on $X^{[k]}$. Although the perturbation technique in [20, Proposition 4.8] can be used to achieve this for every single E_x, for our purpose we will have to extend this technique to prove that all sheaves E_x are slope stable with respect to the same ample class near h_k.

Theorem 2.8 *There exists some ample class $H \in \mathrm{NS}(X^{[k]})$ near h_k, such that E_x is μ_H-stable for all $x \in X$ simultaneously.*

Proof. Proposition 2.7 and [4, Theorem 2.3.1] guarantees that the assumptions in [20, Proposition 4.8] are satisfied for each E_x, hence every E_x is slope stable with respect to some ample class near h_k by [20, Proposition 4.8]. In order to find a single ample class H that is independent of the choice of E_x, we can literally use the entire proof of [20, Proposition 4.8] except that we need to reconstruct the non-empty convex open set U so that $\alpha := h_k^{2k-1}$ is in the closure of U, and for every $\gamma \in U$, E_x is stable with respect to γ for all $x \in X$.

We follow the notations in [7, Definition 3.1]. For each $x \in X$, $\mathrm{SStab}(E_x)$ is a convex closed set containing α. Hence the intersection

$$\overline{U} := \cap_{x\in X} \mathrm{SStab}(E_x)$$

is also a convex closed set containing α. We first claim that [7, Theorem 3.4] holds for all E_x simultaneously; namely, we will show that for any $\beta \in \mathrm{Mov}(X^{[k]})^\circ$ (see [7, Definition 2.1] for the notation), there exists a number $e \in \mathbb{Q}^+$, such that $(\alpha+\varepsilon\beta) \in \cap_{x\in X} \mathrm{Stab}(E_x)$ for any real $\varepsilon \in [0, e]$.

To prove the claim, we first note that the slope $c := \mu_\beta(E_x)$ is independent of the choice of $x \in X$. We redefine the set S in the proof of [7, Theorem 3.4] to be

$$S := \{c_1(F) \mid F \subseteq E_x \text{ for some } x \in X \text{ such that } \mu_\beta(F) \geqslant c\}.$$

Since $E_x \subseteq K$ for all $x \in X$ by (5), we obtain that S is a subset of

$$T := \{c_1(F) \mid F \subseteq K \text{ such that } \mu_\beta(F) \geqslant c\},$$

which is finite by [7, Theorem 2.29], hence S is also finite. We can then use the rest of the proof of [7, Theorem 3.4] literally to conclude the claim.

We then claim that $\overline{U}$ is of full dimension $r := \mathrm{rk}\, N_1(X^{[k]})$. If not, then we have $\alpha \in \overline{U} \subseteq L$ for some hyperplane $L \subset N_1(X^{[k]})_\mathbb{R}$. Since $\mathrm{Mov}(X^{[k]})$ is of full dimension, we can choose some $\beta \in \mathrm{Mov}(X^{[k]})^\circ \setminus L$. It follows that $(\alpha + \varepsilon\beta) \in \overline{U} \setminus L$ for some small $\varepsilon > 0$ by the previous claim and the choice of β. Contradiction.

We define U to be the interior of $\overline{U}$ and claim that U is non-empty. Indeed, since $\overline{U}$ is of full dimension r, we can choose $r + 1$ points of $\overline{U}$ in general positions, which form an r-simplex. By the convexity of $\overline{U}$, the entire simplex is in $\overline{U}$ hence any interior point of the simplex is also an interior point of $\overline{U}$. The convexity of U follows from the convexity of $\overline{U}$. And it is clear from the construction that $\alpha = h_k^{2k-1}$ is in the closure of U. We finally claim that every $\gamma \in U$ is in $\cap_{x\in X} \operatorname{Stab}(E_x)$. If not, suppose that there exists some $\gamma_0 \in U$ and some $x_0 \in X$, such that $\gamma_0 \in \operatorname{SStab}(E_{x_0}) \setminus \operatorname{Stab}(E_{x_0})$; namely, $\mu_{\gamma_0}(F) = \mu_{\gamma_0}(E_{x_0})$ for some proper subsheaf F of E_{x_0}. Since the slope function is linear with respect to the curve class, and $\mu_\alpha(F) < \mu_\alpha(E_{x_0})$ by Proposition 2.7, one can find a hyperplane in $N^1(X^{[k]})_{\mathbb{R}}$ through γ_0, such that $\mu_\gamma(E_{x_0}) - \mu_\gamma(F)$ takes opposite signs for γ in the two open half-spaces separated by the hyperplane. In particular, F destabilizes E_{x_0} in one of the half-spaces. Since U has non-empty intersection with both half-spaces, this contradicts the condition $U \subseteq \operatorname{SStab}(E_x)$. Therefore we have $U \subseteq \cap_{x\in X} \operatorname{Stab}(E_x)$, as desired. □

We give an alternative description of E_x using the integral functor Φ from Example 1.6:

Lemma 2.9 *For each $x \in X$, let I_x be the ideal sheaf of $x \in X$, then $E_x = \Phi(I_x(1))$.*

Proof. We start with the exact sequence

$$0 \longrightarrow E_x \longrightarrow K \longrightarrow I_{S_x} \longrightarrow 0. \tag{10}$$

We note that $I_{S_x} = \Phi(\mathcal{O}_x)$ as $\mathcal{I}_{\mathcal{Z}}$ is flat over X. Furthermore we have $K = \Phi(\mathcal{O}_X(1))$ since $R^i p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) = 0$ for $i = 1, 2$ as this is true for $H^i(I_Z(1))$ for any $[Z] \in X^{[k]}$. These two facts imply that

$$\operatorname{Hom}_{X^{[k]}}(K, I_{S_x}) = \operatorname{Hom}_{X^{[k]}}(\Phi(\mathcal{O}_X(1)), \Phi(\mathcal{O}_x)) \cong \operatorname{Hom}_X(\mathcal{O}_X(1), \mathcal{O}_x) \cong \mathbb{C}$$

by Remark 1.7. Thus the exact sequence (10) is induced by the exact sequence

$$0 \longrightarrow I_x(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

As $K \to I_{S_x}$ is surjective, applying Φ to the last exact sequence shows $E_x = \Phi(I_x(1))$. □

We return to the main result of the section. Let H be an ample class that satisfies Theorem 2.8, and $\mathcal{M}$ the moduli space of μ_H-stable sheaves on $X^{[k]}$ with the same numerical invariants as E_x. Then the universal family $\mathcal{E}$ defines a classifying morphism

$$f \colon X \longrightarrow \mathcal{M}, \quad x \longmapsto [E_x] \tag{11}$$

In fact the morphism f can be described as follows:

Theorem 2.10 *The classifying morphism* (11) *defined by the family $\mathcal{E}$ identifies X with a smooth connected component of $\mathcal{M}$.*

Proof. By [17, Lemma 1.6] we have to prove that f is injective on closed points and that for all $x \in X$ we have $\dim(T_{[E_x]}\mathcal{M}) = 2$.

Now by Lemma 2.9 we know $E_x = \Phi(I_x(1))$, so for $x \neq y$ we find

$$\begin{aligned} \operatorname{Hom}_{X^{[k]}}(E_x, E_y) &= \operatorname{Hom}_{X^{[k]}}(\Phi(I_x(1)), \Phi(I_y(1))) \\ &\cong \operatorname{Hom}_X(I_x(1), I_y(1)) \\ &\cong \operatorname{Hom}_X(\mathcal{O}_x, \mathcal{O}_y) = 0 \end{aligned}$$

by Remark 1.7 again. This implies f is injective on closed points.

A similar computation shows

$$\begin{aligned} \operatorname{Ext}^1_{X^{[k]}}(E_x, E_x) &= \operatorname{Ext}^1_{X^{[k]}}(\Phi(I_x(1)), \Phi(I_x(1))) \\ &\cong \operatorname{Ext}^1_X(I_x(1), I_x(1)) \\ &\cong \operatorname{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_x) \cong T_x X. \end{aligned}$$

Using $T_{[E_x]}\mathcal{M} \cong \operatorname{Ext}^1_{X^{[k]}}(E_x, E_x)$ we thus find $\dim(T_{[E_x]}\mathcal{M}) = 2$ as desired. □

3 K3 surfaces with Picard number two

In this section, we will consider a K3 surface X of Picard number 2, and construct a complete family of stable vector bundles on the Hilbert scheme $X^{[k]}$ for $k \geqslant 2$.

3.1 The K3 surface

In this section we assume X is a K3 surface with

$$\mathrm{NS}(X) = \mathbb{Z}e \oplus \mathbb{Z}f$$

such that $e^2 = -2k$, $f^2 = 0$ and $ef = 2k+1$ for some integer $k \geqslant 2$. The existence of such K3 surfaces is guaranteed by [9, Corollary 14.3.1]. Since $f^2 = 0$, either f or $-f$ is effective. Without loss of generality, we will assume that the divisor class f is effective, after possibly replacing the pair (e, f) by $(-e, -f)$.

In this subsection we collect some helpful properties of X which will be used in the construction of some moduli spaces of stable sheaves in the next section.

Lemma 3.1 *We have $D^2 \geqslant 0$ for all effective divisors on X. Especially there are no smooth curves C on X with $C \cong \mathbb{P}^1$.*

Proof. Any irreducible curve C on S satisfies

$$C^2 = C(C + K_X) = 2p_a(C) - 2 \geqslant -2.$$

So assume $C^2 = -2$ and write $C = me + nf$. Then we have

$$\begin{aligned} C^2 &= (me + nf)^2 = m^2e^2 + 2mnef \\ &= -2km^2 + 2(2k+1)mn \\ &= -2m(km - (2k+1)n). \end{aligned}$$

The equation $C^2 = -2$ translates into $m(km - (2k+1)n) = 1$. This implies $m = \pm 1$ but then one can see that there is no $n \in \mathbb{Z}$ satisfying this equation. □

Lemma 3.2 *The divisor classes $h = e + (2k-1)f$ and $\widehat{h} = (2k)e + (2k-1)f$ are ample.*

Proof. We have

$$\begin{aligned} h^2 &= (e + (2k-1)f)^2 = e^2 + 2(2k-1)ef \\ &= -2k + 2(2k-1)(2k+1) = 8k^2 - 2k - 2. \end{aligned}$$

So $h^2 > 0$ as $k \geqslant 2$. Since also $hf = ef = 2k+1 > 0$ we see that h is ample by the remark after [9, Corollary 8.1.7].

A similar computation shows $\widehat{h}^2 > 0$ and $\widehat{h}f > 0$. □

Lemma 3.3 *Let m and n be integers. If the class $me + nf$ is effective, then $0 \leqslant m \leqslant \frac{2k+1}{k}n$ (thus in particular $n \geqslant 0$). Furthermore $h(me + nf) \geqslant ((2k-1)(2k+1) - k)m$.*

Proof. Let D be an effective divisor with class $me + nf$. Since the claim is additive in m and n, we may assume w.l.o.g. that D is an irreducible curve C.

By Lemma 3.1 we have $C^2 \geqslant 0$. Therefore

$$\begin{aligned} C^2 &= 2m\{-km + (2k+1)n\} \geqslant 0 \\ hC &= (4k^2 - k - 1)m + \{-km + (2k+1)n\} > 0 \end{aligned}$$

which implies $m \geqslant 0$ and $-km + (2k+1)n \geqslant 0$. The last inequality can also be read as

$$(2k+1)n \geqslant km \Leftrightarrow m \leqslant \frac{2k+1}{k}n.$$

Putting everything together shows

$$0 \leqslant m \leqslant \frac{2k+1}{k}n$$

as well as $hC \geqslant ((2k-1)(2k+1) - k)m$. □

Corollary 3.4 *There is a surjective morphism $\pi : X \to \mathbb{P}^1$ such that all fibers are integral curves of arithmetic genus $p_a(C) = 1$, that is X is elliptically fibered.*

Proof. Since $f^2 = 0$ it is known that the linear system $|f|$ induces a surjective map $\pi : X \to \mathbb{P}^1$ with $\pi^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_X(f)$. By the previous lemma the class f cannot be the sum of two effective divisors, hence all fibers C of π are integral and have $p_a(C) = 1$. □

Lemma 3.5 *Let $[Z] \in X^{[k]}$. Assume R is a torsion quotient of $I_Z(e)$ with $c_1(R) = nf$ for some $n \geqslant 0$, then $H^1(R) = 0$.*

Proof. The quotient defines the following exact sequence:

$$0 \longrightarrow K \longrightarrow I_Z(e) \longrightarrow R \longrightarrow 0.$$

Now K is torsion free of rank one, so its double dual K^{**} is locally free of rank one and the natural map $K \to K^{**}$ is injective and the cokernel T has finite support. Especially $c_1(T) = 0$ so

$$c_1(K^{**}) = c_1(K) = c_1(I_Z(e)) - c_1(R) = e - nf$$

and thus $K^{**} \cong \mathcal{O}_X(e - nf)$. The embedding $K \hookrightarrow I_Z(e)$ induces an embedding

$$K^{**} \cong \mathcal{O}_X(e - nf) \hookrightarrow \mathcal{O}_X(e).$$

This embedding is given by a global section of $\mathcal{O}_X(nf)$, that is by an effective divisor $D = \sum_i a_i C_i$ with class nf.

This global section is the pullback along the elliptic fibration π of a global section of $\mathcal{O}_{\mathbb{P}^1}(n)$, with corresponding effective divisor $\sum_i a_i z_i$ on $\mathbb{P}^1$, here $C_i = \pi^{-1}(z_i)$.

Denote by $D \subset X$ also the corresponding closed subscheme (which maybe non-reduced, if $a_i \geqslant 2$ for some i). We get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & \mathcal{O}_X(e-nf) & \longrightarrow & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \alpha & & \\
0 & \longrightarrow & I_Z(e) & \longrightarrow & \mathcal{O}_X(e) & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & R & \xrightarrow{\beta} & \mathcal{O}_D(e) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The snake lemma gives an exact sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow R \xrightarrow{\beta} \mathcal{O}_D(e) \longrightarrow \operatorname{coker}(\alpha) \longrightarrow 0.$$

Let $R' \subset \mathcal{O}_D(e)$ be the image of β. Since the torsion sheaf $\mathcal{O}_{\sum_i a_i z_i}$ on $\mathbb{P}^1$ has a composition series by skyscraper sheaves $\mathcal{O}_{z_i}$ as composition factors, $\mathcal{O}_D$ has a composition series with composition factors $\mathcal{O}_{C_i}$, thus $\mathcal{O}_D(e)$ has a composition series with composition factors $\mathcal{O}_{C_i}(e)$. The latter is a line bundle of degree

$$e \cdot C_i = ef = 2k + 1$$

on C_i. The quotient $\mathcal{O}_D(e)/R'$ is isomorphic to $\operatorname{coker}(\alpha)$, that is to a quotient Q of $\mathcal{O}_Z$. By intersecting with R' we get a composition series for R' with composition factors which are kernels of a surjection $\mathcal{O}_{C_i}(e) \twoheadrightarrow Q'$ with Q' of length $\leqslant k$. Thus we have exact sequences:

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{C_i}(e) \longrightarrow Q' \longrightarrow 0,$$

with a torsion free sheaf L of rank one on the integral projective curve C_i of arithmetic genus one. Using $\chi(\mathcal{O}_{C_i}) = 0$ and

$$\chi(L) = \chi(\mathcal{O}_{C_i}(e)) - \chi(Q') \geqslant k + 1,$$

gives

$$\deg(\mathcal{O}_{C_i}(e)) \geqslant \deg(L) \geqslant k + 1.$$

By [6, Proposition 4.6.] all of these composition factors have trivial H^1. By constructing short exact sequences out of the composition series and using the induced exact sequences for H^1, it follows

$$H^1(R') = 0.$$

As $\ker(\beta) = \ker(\alpha) \subseteq T$ has finite support, we also have $H^1(\ker(\beta)) = 0$. Hence

$$H^1(R) = 0.$$

□

3.2 The construction of a universal family

In this subsection we want to generalize [16, Theorem 1.2]. Let h be the ample line bundle defined in Lemma 3.2, and $v = (2k-1, h, 2k) \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ for any integer $k \geqslant 2$. We immediately have the following result:

Lemma 3.6 *The moduli space $M_h(v)$ of μ_h-stable sheaves on X with Mukai vector v is a smooth projective variety of dimension $2k$ and a fine moduli space, and every point $[E] \in M_h(v)$ represents a locally free sheaf.*

Proof. We first observe by [10, Lemma 1.2.7] that every μ_h-semistable sheaf E with $v(E) = v$ is μ_h-stable since $\gcd(2k-1, h^2) = 1$. Thus $M_h(v)$ is a smooth projective variety. We compute:

$$\dim(M_h(v)) = v^2 + 2 = (8k^2 - 2k - 2) - 2(2k-1)(2k) + 2 = 2k.$$

Furthermore $v' = (2k-1, h, a)$ with $a \geqslant 2k+1$ satisfies

$$v'^2 + 2 = h^2 - 2a(2k-1) + 2 \leqslant (8k^2 - 2k - 2) - 2(2k-1)(2k+1) + 2 = 2 - 2k < 0,$$

so again every point $[E]$ in $M_h(v)$ is locally free. The condition $\gcd(2k-1, h^2) = 1$ also implies that $M_h(v)$ is a fine moduli space. □

In the following discussion, we will explicitly construct a universal family for the moduli space $M_h(v)$. We first define some integral functors. For any line bundle L on X, we define

$$M_L : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X); \quad (-) \longmapsto (-) \otimes L.$$

Then we consider the composition

$$\Theta := M_{\mathcal{O}_X(f)} \circ T^{-1}_{\mathcal{O}_X} \circ M_{\mathcal{O}_X(e)} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X), \tag{12}$$

where $T^{-1}_{\mathcal{O}_X}$ is the inverse of the spherical twist induced by $\mathcal{O}_X$. It is clear that Θ is an autoequivalence of $\mathrm{D^b}(X)$ hence a Fourier-Mukai transform. We denote the corresponding kernel by $\mathcal{P} \in \mathrm{D^b}(X \times X)$. By Remark 1.5, we have an explicit description of $\mathcal{P}$ by the exact triangle

$$\mathcal{P} \longrightarrow \Delta_* \mathcal{O}_X(e+f) \longrightarrow \mathcal{O}_X(e) \boxtimes \mathcal{O}_X(f)[2] \longrightarrow \mathcal{P}[1], \tag{13}$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding. The kernel $\mathcal{P}$ also defines a Fourier-Mukai transform in the opposite direction, which we denote by

$$\widehat{\Theta} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X).$$

Since the kernel of each composition factor in (12), viewed as an object in $\mathrm{D^b}(X \times X)$, remains the same under the permutation of the two copies of X, it follows that

$$\widehat{\Theta} = M_{\mathcal{O}_X(e)} \circ T^{-1}_{\mathcal{O}_X} \circ M_{\mathcal{O}_X(f)}. \tag{14}$$

For any $[Z] \in X^{[k]}$, we apply Θ on the ideal sheaf I_Z and define

$$E_Z := \Theta(I_Z).$$

A priori E_Z is a derived object on X, but we can show the following:

Theorem 3.7 *E_Z is μ_h-stable locally free sheaf with Mukai vector $v(E_Z) = (2k-1, h, 2k)$.*

Proof. First of all, by (12) and [8, Lemma 8.12], a standard computation of the cohomological Fourier-Mukai transform shows that

$$v(E_Z) = (2k-1, h, 2k).$$

Moreover, by (13) and the fact that $T^{-1}_{\mathcal{O}_X}(\mathcal{O}_X) = \mathcal{O}_X[1]$, we obtain an exact triangle

$$E_Z \longrightarrow I_Z(e+f) \longrightarrow H^*(I_Z(e)) \otimes \mathcal{O}_X(f)[2] \longrightarrow E_Z[1]. \tag{15}$$

In order to compute $H^*(I_Z(e))$, we observe by Lemma 3.3 that

$$h^0(\mathcal{O}_X(e)) = 0 \quad \text{and} \quad h^2(\mathcal{O}_X(e)) = h^0(\mathcal{O}_X(-e)) = 0. \tag{16}$$

It follows by a long exact sequence of cohomology groups that

$$h^0(I_Z(e)) = h^2(I_Z(e)) = 0.$$

Therefore the exact triangle (15) reduces to the short exact sequence

$$0 \longrightarrow H^1(I_Z(e)) \otimes \mathcal{O}_X(f) \longrightarrow E_Z \longrightarrow I_Z(e+f) \longrightarrow 0, \tag{17}$$

where $\dim H^1(I_Z(e)) = \mathrm{rk}(E_Z) - 1 = 2k-2$. For the convenience of analyzing the stability of E_Z, we rewrite the above exact triangle as

$$0 \longrightarrow \mathcal{O}_X^{\oplus(2k-2)} \longrightarrow E_Z(-f) \longrightarrow I_Z(e) \longrightarrow 0.$$

Furthermore, we observe that $\mathcal{O}_X(f) = \Theta(\mathcal{O}_X(-e)[-1])$. Since Θ is an equivalence, we have

$$\mathrm{Hom}(E_Z(-f), \mathcal{O}_X) = \mathrm{Hom}(E_Z, \mathcal{O}_X(f)) = \mathrm{Hom}(I_Z, \mathcal{O}_X(-e)[-1]) = 0.$$

We are now ready to prove that E_Z, or rather $E_Z(-f)$, is μ_h-stable. We first have

$$\mu_h(E_Z(-f)) = \frac{eh}{2k-1} = \frac{-2k + (2k-1)(2k+1)}{2k-1} = 2k+1 - \frac{2k}{2k-1} > 0.$$

Pick a torsion free quotient F of $E_Z(-f)$ with $1 \leqslant \mathrm{rk}(F) \leqslant 2k-2$. We have

$$E_Z(-f) \longrightarrow F \longrightarrow 0$$

with $\mathrm{Hom}(F, \mathcal{O}_X) \hookrightarrow \mathrm{Hom}(E_Z(-f), \mathcal{O}_X) = 0$.

We want to show that we always have $\mu_h(F) > \mu_h(E_Z(-f))$. For this, define the torsion free sheaf F_0 as the image of the composition

$$\mathcal{O}_X^{\oplus(2k-2)} \hookrightarrow E_Z(-f) \longrightarrow F.$$

We get a surjection

$$\mathcal{O}_X^{\oplus(2k-2)} \longrightarrow F_0 \longrightarrow 0.$$

This implies that $c_1(F_0)$ is effective and we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{\oplus(2k-2)} & \longrightarrow & E_Z(-f) & \longrightarrow & I_Z(e) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F & \longrightarrow & F_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{18}$$

Due to the diagram $\mathrm{rk}(F_1) \in \{0,1\}$.

Case 1: $\mathrm{rk}(F_1) = 1$. Then $\mathrm{rk}(F_0) = \mathrm{rk}(F) - 1$ and $F_1 \cong I_Z(e)$. We conclude

$$c_1(F) = c_1(F_0) + c_1(I_Z(e)) \Rightarrow c_1(F) = c_1(F_0) + e.$$

Using this we find:

$$\mu_h(F) = \frac{c_1(F)h}{\mathrm{rk}(F)} = \underbrace{\frac{c_1(F_0)h}{\mathrm{rk}(F)}}_{\geqslant 0} + \frac{eh}{\mathrm{rk}(F)} > \frac{eh}{2k-1} = \mu_h(E_Z(-f)).$$

So we indeed have $\mu_h(F) > \mu_h(E_x(-f))$.

Case 2: $\mathrm{rk}(F_1) = 0$. Now $\mathrm{rk}(F_0) = \mathrm{rk}(F)$. Write $c_1(F) = me + nf$. Since $c_1(F_0)$ and $c_1(F_1)$ are effective, so is their sum $c_1(F)$, which by Lemma 3.3 implies, that $m \geqslant 0$ as well as

$$\mu_h(F) = \frac{(me+nf)h}{\mathrm{rk}(F)} \geqslant \frac{m((2k-1)(2k+1)-k)}{\mathrm{rk}(F)} \geqslant m(2k+1-\frac{k}{2k-1}).$$

For $m \geqslant 1$ we have

$$\begin{aligned}\mu_h(F) &\geqslant m(2k+1-\frac{k}{2k-1}) \\ &\geqslant 2k+1-\frac{k}{2k-1} \\ &> 2k+1-\frac{2k}{2k-1} = \mu_h(E_Z(-f))\end{aligned}$$

So only the case $m = 0$ remains, i.e. $c_1(F) = nf$. We have

$$\mu_h(F) = \frac{n(2k+1)}{\mathrm{rk}(F)}.$$

If we can prove $n \geqslant \mathrm{rk}(F)$ we are done since then

$$\mu_h(F) \geqslant 2k+1 > 2k+1-\frac{2k}{2k-1} = \mu_h(E_Z(-f)).$$

As $c_1(F) = nf$ is the sum of the two effective divisors $c_1(F_0)$ and $c_1(F_1)$, it follows from Lemma 3.3 that $c_1(F_0) = n_0 f$ and $c_1(F_1) = n_1 f$ with $n_0, n_1 \geqslant 0$ and $n_0 + n_1 = n$.

By Lemma 3.5 we have $H^1(F_1) = 0$ which implies $\mathrm{Ext}^1(F_1, \mathcal{O}_X) = 0$ using Serre duality. So the restriction map

$$\mathrm{Hom}(F, \mathcal{O}_X) \to \mathrm{Hom}(F_0, \mathcal{O}_X)$$

surjective. But we know $\mathrm{Hom}(F, \mathcal{O}_X) = 0$. So

$$\mathrm{Hom}(F_0, \mathcal{O}_X) = 0. \tag{19}$$

Using the elliptic fibration $\pi : X \to \mathbb{P}^1$ we have:

$$h^0(\det(F_0)) = h^0(\mathcal{O}_X(n_0 f)) = n_0 + 1. \tag{20}$$

Now there is a trivial sub-bundle in $\mathcal{O}_X^{\oplus(2k-2)}$ of rank $\mathrm{rk}(F)+1$ such that

$$\mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)} \xrightarrow{\ \varphi\ } F_0$$

is surjective outside a finite subset of X by [3, Lemma 4.60].

Define $R := \mathrm{coker}(\varphi)$. Then there is the exact sequence:

$$0 \longrightarrow F_0' \longrightarrow F_0 \longrightarrow R \longrightarrow 0.$$

As R has finite support, we get:

$$\det(F_0) = \det(F_0') \text{ as well as } H^2(F_0') \cong H^2(F_0).$$

We also have the exact sequence

$$0 \longrightarrow \det(F_0)^{-1} \longrightarrow \mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)} \longrightarrow F_0' \longrightarrow 0.$$

The end of the induced long cohomology sequence gives:

$$H^1(F_0') \longrightarrow H^2(\det(F_0)^{-1}) \longrightarrow H^2(\mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)}) \longrightarrow H^2(F_0') \longrightarrow 0. \tag{21}$$

It follows from (19) by Serre duality that

$$H^2(F_0') \cong H^2(F_0) \cong \mathrm{Hom}(F_0, \mathcal{O}_X)^\vee = 0.$$

Since $H^2(F_0') = 0$, we apply Serre duality again and obtain from (21) that

$$0 \longrightarrow H^0(\mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)}) \longrightarrow H^0(\det(F_0)).$$

We conclude

$$h^0(\det(F_0)) \geqslant \mathrm{rk}(F) + 1.$$

Using this inequality together with (20) we get:

$$n_0 + 1 = h^0(\det(F_0)) \geqslant \mathrm{rk}(F) + 1 \Rightarrow n_0 \geqslant \mathrm{rk}(F) \Rightarrow n \geqslant \mathrm{rk}(F).$$

We obtain the desired inequality between n and $\mathrm{rk}(F)$, hence $E_Z(-f)$ is stable, and so is E_Z. It then follows by Lemma 3.6 that E_Z is locally free. □

We want to globalize the previous construction. For this we denote the universal closed subscheme of length n by $\mathcal{Z} \subset X \times X^{[k]}$, and the universal ideal sheaf by $\mathcal{I}_{\mathcal{Z}}$. As a kernel, $\mathcal{I}_{\mathcal{Z}}$ induces a pair of integral functors (in opposite directions):

$$\Phi : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X^{[k]}) \ \text{ and } \ \widehat{\Phi} : \mathrm{D}^{\mathrm{b}}(X^{[k]}) \longrightarrow \mathrm{D}^{\mathrm{b}}(X).$$

Here Φ is a $\mathbb{P}^{k-1}$-functor, see Example 1.6.

The composition of the integral functors

$$\Theta \circ \widehat{\Phi} : \mathrm{D}^{\mathrm{b}}(X^{[k]}) \longrightarrow \mathrm{D}^{\mathrm{b}}(X)$$

is still an integral functor, whose kernel $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X^{[k]} \times X)$ can be computed from $\mathcal{P}$ and $\mathcal{I}_{\mathcal{Z}}$ explicitly. More precisely, let π_{12}, π_{23} and π_{13} be projections from $X^{[k]} \times X \times X$ to each pair of factors, then

$$\mathcal{E} = R\pi_{13*}(\pi_{12}^* \mathcal{I}_{\mathcal{Z}} \otimes \pi_{23}^* \mathcal{P});$$

see [8, Proposition 5.10]. We have the following property about $\mathcal{E}$:

Proposition 3.8 *$\mathcal{E}$ is a locally free sheaf on $X^{[k]} \times X$ such that $\mathcal{E}|_{\{[Z]\} \times X} \cong E_Z$ for any $[Z] \in X^{[k]}$.*

Proof. For any $[Z] \in X^{[k]}$, the derived pullback of $\mathcal{E}$ to the fiber $\{[Z]\} \times X$ can be computed by

$$(\Theta \circ \widehat{\Phi})(\mathcal{O}_{[Z]}) \cong \Theta(I_Z) = E_Z,$$

which is a locally free sheaf by Theorem 3.7. It follows that $\mathcal{E}$ is a sheaf which is flat over $X^{[k]}$ by [8, Lemma 3.31], and locally free by [10, Lemma 2.1.7]. □

In fact, $\mathcal{E}$ is a universal family for the fine moduli space $M_h(v)$. More precisely, we have

Corollary 3.9 *The family $\mathcal{E}$ induces an isomorphism $X^{[k]} \cong M_h(v)$.*

Proof. $\mathcal{E}$ induces a classifying morphism

$$\varphi : X^{[k]} \longrightarrow M_h(v); \quad [Z] \longmapsto [E_Z].$$

Since Θ is an equivalence, we have $E_Z \ncong E_{Z'}$ for $[Z] \neq [Z']$, hence φ is injective, hence it is an open embedding since $X^{[k]}$ and $M_h(v)$ are both of dimension $2k$. But $X^{[k]}$ is projective, so φ is also closed. Since $X^{[k]}$ and $M_h(v)$ are both irreducible, φ must be an isomorphism. □

Remark 3.10 Although it is not strictly required in our following discussion, the universal family $\mathcal{E}$ can in fact be given in a more explicit form similar to (17). To globalize the construction in Theorem 3.7, we apply the functor $R\pi_{13*}(\pi_{12}^*\mathcal{I}_{\mathcal{Z}} \otimes \pi_{23}^*(-))$ to (13) and obtain

$$\mathcal{E} \longrightarrow R\pi_{13*}(\pi_{12}^*\mathcal{I}_{\mathcal{Z}} \otimes \pi_{23}^*\Delta_* \mathcal{O}_X(e+f)) \longrightarrow R\pi_{13*}(\pi_{12}^*\mathcal{I}_{\mathcal{Z}} \otimes \pi_2^* \mathcal{O}_X(e) \otimes \pi_3^* \mathcal{O}_X(f))[2] \longrightarrow \mathcal{E}[1].$$

We denote the projections from $X^{[k]} \times X$ to the two factors by p and q respectively. Then a simple calculation reduces the above exact triangle to

$$\mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e+f) \longrightarrow Rp_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e)) \boxtimes \mathcal{O}_X(f)[2] \longrightarrow \mathcal{E}[1].$$

For the consistency with the following discussion, we denote

$$\mathcal{H} \coloneqq Rp_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e))[1] = \Phi(\mathcal{O}_X(e))[1].$$

We will prove in Lemma 3.11 that $\mathcal{H}$ is in fact a sheaf. Therefore the exact triangle reduces to

$$0 \longrightarrow \mathcal{H} \boxtimes \mathcal{O}_X(f) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e+f) \longrightarrow 0.$$

3.3 The wrong-way fibers

In this subsection we study the wrong-way fibers of $\mathcal{E}$. For any $x \in X$, we define the corresponding wrong-way fiber to be

$$E_x := \mathcal{E}|_{X^{[k]}\times\{x\}},$$

which is locally free of rank $2k-1$. As an alternative description, we consider the composition

$$\Phi \circ \widehat{\Theta} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X^{[k]}),$$

which is also an integral functor with kernel $\mathcal{E}$, in the direction opposite to $\Theta \circ \widehat{\Phi}$. Then we have

$$E_x = (\Phi \circ \widehat{\Theta})(\mathcal{O}_x).$$

The following result gives a concrete description of E_x:

Lemma 3.11 *The locally free sheaf E_x fits in an exact sequence of sheaves*

$$0 \longrightarrow \mathcal{H} \longrightarrow E_x \longrightarrow I_{S_x} \longrightarrow 0, \tag{22}$$

where

$$\mathcal{H} := \Phi(\mathcal{O}_X(e))[1]$$

is locally free, and I_{S_x} is the ideal sheaf of

$$S_x := \left\{ [Z] \in X^{[k]} \,\middle|\, x \in \mathrm{supp}(Z) \right\} \subset X^{[k]}.$$

Proof. We write $F_x := \widehat{\Theta}(\mathcal{O}_x)$, then $E_x = \Phi(F_x)$. By (14) we have $F_x = T^{-1}_{\mathcal{O}_X}(\mathcal{O}_x) \otimes \mathcal{O}_X(e)$. By applying the inverse spherical functor $T^{-1}_{\mathcal{O}_X}$ on the exact sequence

$$0 \longrightarrow I_x \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_x \longrightarrow 0$$

we obtain an exact triangle

$$T^{-1}_{\mathcal{O}_X}(I_x) \longrightarrow T^{-1}_{\mathcal{O}_X}(\mathcal{O}_X) \longrightarrow T^{-1}_{\mathcal{O}_X}(\mathcal{O}_x) \longrightarrow T^{-1}_{\mathcal{O}_X}(I_x)[1].$$

Since $T_{\mathcal{O}_X}(\mathcal{O}_X) = \mathcal{O}_X[-1]$ and $T_{\mathcal{O}_X}(\mathcal{O}_x) = I_x[1]$, the above triangle becomes

$$\mathcal{O}_x[-1] \longrightarrow \mathcal{O}_X(e)[1] \longrightarrow F_x \longrightarrow \mathcal{O}_x\,.$$

Since $\Phi(\mathcal{O}_x) = I_{S_x}$, we further apply the integral functor Φ to obtain the exact triangle

$$I_{S_x}[-1] \longrightarrow \mathcal{H} \longrightarrow E_x \longrightarrow I_{S_x}, \tag{23}$$

where $\mathcal{H} = \Phi(\mathcal{O}_X(e))[1]$. To compute $\mathcal{H}$, we observe that the short exact sequence of kernels

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{X^{[k]}\times X} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0$$

induces an exact triangle

$$\Phi(\mathcal{O}_X(e)) \longrightarrow H^*(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^{[k]}} \longrightarrow \mathcal{O}_X(e)^{[k]} \longrightarrow \Phi(\mathcal{O}_X(e))[1]. \tag{24}$$

Since $H^i(\mathcal{O}_X(e)) = 0$ for $i \neq 1$ by (16), the exact triangle (24) reduces to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(e)^{[k]} \longrightarrow \mathcal{H} \longrightarrow H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^{[k]}} \longrightarrow 0,$$

which in particular implies that $\mathcal{H}$ is a locally free sheaf. It follows that the exact triangle (23) reduces to the short exact sequence (22). □

We will require a technical result in the proof of the stability. For this purpose, we define

$$I^kX := (X^k \times_{S^kX} X^{[k]})_{\mathrm{red}}$$

to be Haiman's isospectral Hilbert scheme, and denote its projections to both factors by p and q respectively. Then the derived McKay correspondence

$$\Psi := (-)^{\mathfrak{S}_k} \circ q_* \circ Lp^* : \mathrm{D^b}(X^k)^{\mathfrak{S}_k} \longrightarrow \mathrm{D^b}(X^{[k]})$$

is an equivalence, and so is $\Psi^{-1} : \mathrm{D^b}(X^{[k]}) \to \mathrm{D^b}(X^k)^{\mathfrak{S}_k}$. We have

Lemma 3.12 *For any coherent sheaf F on $X^{[k]}$, if $\Psi^{-1}(F)$ is a reflexive sheaf, then*

$$\Psi^{-1}(F) = (F)_{X^k}.$$

Proof. We follow the above notation to denote

$$I^k X_\circ := X_\circ^k \times_{S^k X_\circ} X_\circ^{[k]},$$

then we have the commutative diagram

$$\begin{array}{ccccc} X_\circ^{[k]} & \xleftarrow{q_\circ} & I^k X_\circ & \xrightarrow{p_\circ} & X_\circ^k \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle j} \\ X^{[k]} & \xleftarrow{q} & I^k X & \xrightarrow{p} & X^k, \end{array}$$

where α, β, j and $q_\circ$ are étale morphisms, and $p_\circ$ is an isomorphism. We also have

$$\Psi^{-1} \cong Rp_* \circ q^!,$$
$$(-)_{X^k} = j_* \circ \overline{\sigma}_\circ^* \circ \alpha^*,$$

where the first equation is due to the fact that Ψ^{-1} is the right adjoint of Ψ. It follows that

$$\begin{aligned} j^* \circ \Psi^{-1} &\cong j^* \circ Rp_* \circ q^! \cong p_{\circ *} \circ \beta^* \circ q^! \\ &\cong p_{\circ *} \circ \beta^! \circ q^! \cong p_{\circ *} \circ q_\circ^! \circ \alpha^! \\ &\cong p_{\circ *} \circ q_\circ^* \circ \alpha^* \cong \overline{\sigma}_\circ^* \circ \alpha^*. \end{aligned}$$

Therefore we have

$$j_* \circ j^* \circ \Psi^{-1} \cong (-)_{X^k}.$$

Since $\Delta = X^k \setminus X_\circ^k$ is of codimension 2, if $\Psi^{-1}(F)$ is a reflexive sheaf, then we have

$$\Psi^{-1}(F) \cong j_* \circ j^* \circ \Psi^{-1}(F) \cong (F)_{X^k}$$

as desired. □

Lemma 3.13 *The sheaf $(\mathcal{H})_{X^k}$ fits in an exact sequence of $\mathfrak{S}_k$-invariant locally free sheaves*

$$0 \longrightarrow \bigoplus_{i=1}^{k} q_i^* \mathcal{O}_X(e) \longrightarrow (\mathcal{H})_{X^k} \longrightarrow H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k} \longrightarrow 0. \tag{25}$$

Moreover, every $\mathfrak{S}_k$-invariant global section of $(\mathcal{H})_{X^k}$ vanishes; namely $H^0((\mathcal{H})_{X^k})^{\mathfrak{S}_k} = 0$.

Proof. By [12, Theorem 3.6], the composition $\Psi^{-1} \circ \Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(X^n)^{\mathfrak{S}_k}$ agrees with the truncated universal ideal functor defined in [13, Definition 5.1], therefore we have an exact triangle

$$(\Psi^{-1} \circ \Phi)(\mathcal{O}_X(e)) \longrightarrow H^*(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k} \xrightarrow{\delta} \bigoplus_{i=1}^{k} q_i^* \mathcal{O}_X(e) \longrightarrow (\Psi^{-1} \circ \Phi)(\mathcal{O}_X(e))[1], \tag{26}$$

where each component of δ is an evaluation map. Since $\mathcal{H}$ is a locally free sheaf by Lemma 3.11, it follows by Lemma 3.12 that $\Psi^{-1}(\mathcal{H}) = (\mathcal{H})_{X^k}$. Hence

$$(\Psi^{-1} \circ \Phi)(\mathcal{O}_X(e)) = \Psi^{-1}(\mathcal{H})[-1] = (\mathcal{H})_{X^k}[-1].$$

Together with (16), the exact triangle (26) becomes the short exact sequence (25), which is the universal equivariant extension of $\mathcal{O}_{X^k}$ by $\bigoplus_{i=1}^k q_i^* \mathcal{O}_X(e)$ since δ is a collection of evaluation maps. Therefore its induced connecting map in the long exact sequence of cohomology groups

$$H^0\left(H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k}\right)^{\mathfrak{S}_k} \longrightarrow H^1\left(\bigoplus_{i=1}^{k} q_i^* \mathcal{O}_X(e)\right)^{\mathfrak{S}_k}$$

is naturally an isomorphism, which implies $H^0((\mathcal{H})_{X^k})^{\mathfrak{S}_k} = 0$. □

We will eventually prove the stability of E_x with respect to some ample class $H \in \mathrm{NS}(X^{[k]})$. Similar to the previous section we have

$$\mathrm{NS}(X^{[k]}) = \mathbb{Z}e_k \oplus \mathbb{Z}f_k \oplus \mathbb{Z}\delta.$$

For any $l \in \mathrm{NS}(X)$ and any ample class $h \in \mathrm{NS}(X)$ we have the intersection numbers

$$\begin{aligned} l_k h_k^{2k-1} &= \frac{(2k-1)!}{(k-1)!2^{k-1}}(lh)(h^2)^{k-1}, \\ \delta h_k^{2k-1} &= 0 \end{aligned}$$

by [21, Lemma 1.10]. Moreover, by Lemma 3.11 and [21, Lemma 1.5] we also have

$$c_1(E_x) = c_1(\mathcal{H}) = c_1(\mathcal{O}_X(e)^{[k]}) = e_k - \delta.$$

It follows by the above formulas that for any ample class $h \in \mathrm{NS}(X)$, we have

$$c_1(E_x)h_k^{2k-1} = \frac{(2k-1)!}{(k-1)!2^{k-1}}(eh)(h^2)^{k-1}.$$

However, $\mathcal{O}_X(e)^{[k]}$ is a subsheaf of E_x with the same c_1. For E_x to be μ_{h_k}-stable, it is necessary to have $eh < 0$ since $h^2 > 0$. An easy computation shows that this condition cannot be fulfilled by the class $h = e + (2k-1)f$ from Lemma 3.2, so we cannot hope that E_x is μ-stable with respect to the class h_k induced by this h. However, for the class $\widehat{h} = (2k)e + (2k-1)f$ from Lemma 3.2, we do have

$$\begin{aligned} e\widehat{h} &= (2k)e^2 + (2k-1)ef \\ &= -(4k^2) + (4k^2-1) = -1. \end{aligned}$$

Indeed, in the rest of this subsection we will prove that E_x is μ-stable with respect to $\widehat{h}_k$. We use the same notation as in Section 2.2 and also need the following formula: assume F is a coherent sheaf on X^k with $\mathfrak{S}_k$-invariant Chern class

$$c_1(F) = \sum_{i=1}^{k} q_i^* c$$

where $c \in \mathrm{NS}(X)$, then the intersection number

$$c_1(F)\widehat{h}_{X^k}^{2k-1} = a_k(c \cdot \widehat{h})(\widehat{h}^2)^{k-1}$$

where $a_k = \frac{k(2k-1)!}{2^{k-1}}$; see [21, Lemma 1.10]. The main result of this subsection is the following

Proposition 3.14 *E_x is μ-stable with respect to $\widehat{h}_k$.*

Proof. Assume that F is a reflexive subsheaf of E_x of rank $1 \leqslant r \leqslant 2k-2$. We need to show that $\mu_{\widehat{h}_k}(F) < \mu_{\widehat{h}_k}(E_x)$. By [20, Lemma 1.2], it suffices to check that

$$\mu_{\widehat{h}_{X^k}}((F)_{X^k}) < \mu_{\widehat{h}_{X^k}}((E_x)_{X^k}),$$

where $(F)_{X^k}$ is an $\mathfrak{S}_k$-invariant subsheaf of $(E_x)_{X^k}$.

We apply the functor $j_*(\overline{\sigma}_{k,\circ}^*((-)_\circ))$ to (22). Since the functor is left exact, together with [20, Lemma 1.1] we obtain that

$$0 \longrightarrow (\mathcal{H})_{X^k} \longrightarrow (E_x)_{X^k} \longrightarrow (I_{S_x})_{X^k} \longrightarrow Q \longrightarrow 0, \tag{27}$$

such that $\mathrm{supp}(Q) \subseteq \Delta$, where $\Delta = X^k \setminus X_\circ^k$ is the big diagonal. It is also clear that

$$\overline{\sigma}_{k,\circ}^*((I_{S_x})_\circ) = \left(\bigotimes_{i=1}^{k} q_i^* I_x\right)\Bigg|_{X^k \setminus \Delta}.$$

Since Δ is of codimension 2 in X^k, we have that $c_1((I_{S_x})_{X^k}) = 0$. It follows that

$$c_1((E_x)_{X^k}) = c_1((\mathcal{H})_{X^k}).$$

Moreover, we have by (25) that

$$c_1((\mathcal{H})_{X^k}) = \sum_{i=1}^{k} q_i^* e.$$

Therefore

$$\begin{aligned} c_1((E_x)_{X^k})\widehat{h}_{X^k}^{2k-1} &= c_1((\mathcal{H})_{X^k})\widehat{h}_{X^k}^{2k-1} \\ &= a_k(e\widehat{h})(\widehat{h}^2)^{k-1} \\ &= a_k(-1)(\widehat{h}^2)^{k-1}. \end{aligned}$$

Since $(F)_{X^k}$ is $\mathfrak{S}_k$-invariant, we have $c_1((F)_{X^k}) = \sum_{i=1}^{k} q_i^* c$ for some $c \in \mathrm{NS}(X)$, and

$$c_1((F)_{X^k})\widehat{h}_{X^k}^{2k-1} = a_k(c \cdot \widehat{h})(\widehat{h}^2)^{k-1}.$$

We have the following two cases:

If $c \cdot \widehat{h} \leqslant -1$, then we have

$$c_1((F)_{X^k})\widehat{h}_{X^k}^{2k-1} \leqslant c_1((E_x)_{X^k})\widehat{h}_{X^k}^{2k-1} < 0.$$

Since $\mathrm{rk}((F)_{X^k}) < \mathrm{rk}((E_x)_{X^k})$, it follows that

$$\mu_{\widehat{h}_{X^k}}((F)_{X^k}) < \mu_{\widehat{h}_{X^k}}((E_x)_{X^k}).$$

If $c \cdot \widehat{h} \geqslant 0$, then $c_1((F)_{X^k})\widehat{h}_{X^k}^{2k-1} \geqslant 0$.

We choose a (not necessarily $\mathfrak{S}_k$-invariant) non-zero $\mu_{\widehat{h}_{X^k}}$-stable reflexive subsheaf of maximal slope $F' \subseteq (F)_{X^k}$, then $\mu_{\widehat{h}_{X^k}}(F') \geqslant 0$. However $q_i^* \mathcal{O}_X(e)$ is $\mu_{\widehat{h}_{X^k}}$-stable for $i = 1, \ldots, k$, and

$$c_1(q_i^* \mathcal{O}_X(e))\widehat{h}_{X^k}^{2k-1} = a_k(e\widehat{h})(\widehat{h}^2)^{k-1} = a_k(-1)(\widehat{h}^2)^{k-1} < 0.$$

Hence the only map from F' to $q_i^* \mathcal{O}_X(e)$ is zero.

By (27) we obtain a morphism $F' \stackrel{\alpha}{\to} (I_{S_x})_{X^k}$. It is clear that $(I_{S_x})_{X^k}$ is torsion free, so it is a subsheaf of its double dual $(I_{S_x})_{X^k}^{\vee\vee}$. We also note that the restriction of $(I_{S_x})_{X^k}$ on $X^k \setminus (\Delta \cup q_1^{-1}(\{x\}) \cup \cdots \cup q_k^{-1}(\{x\}))$ is the trivial line bundle, hence

$$(I_{S_x})_{X^k}^{\vee\vee} = \mathcal{O}_{X^k}.$$

Therefore we have

$$F' \stackrel{\alpha}{\to} (I_{S_x})_{X^k} \hookrightarrow \mathcal{O}_{X^k}.$$

If $\alpha \neq 0$, then the composition of both maps is non-zero, hence the stability forces

$$\mu_{\widehat{h}_{X^k}}(F') = 0 = \mu_{\widehat{h}_{X^k}}(\mathcal{O}_{X^k}).$$

Since F' is reflexive, the composition must be the identity map. Since $(I_{S_x})_{X^k} \neq \mathcal{O}_{X^k}$ this is a contradiction. It follows that $\alpha = 0$, which implies by (27) that F' is a subsheaf of $(\mathcal{H})_{X^k}$. By (25) and the above discussion, we can furthermore conclude that F' is isomorphic to a subsheaf of the trivial bundle $H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k}$. The stability forces again that

$$\mu_{\widehat{h}_{X^k}}(F') = 0 = \mu_{\widehat{h}_{X^k}}(\mathcal{O}_{X^k})$$

and $F' \cong \mathcal{O}_{X^k}$. Moreover, since all global sections of the trivial bundle $H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k}$ in (25) are invariant under the permutation of $\mathfrak{S}_k$, we conclude that F' itself is also $\mathfrak{S}_k$-invariant, which gives a non-trivial $\mathfrak{S}_k$-invariant global section of $\mathcal{H}_{X^k}$. This contradicts Lemma 3.13, therefore $(E_x)_{X^k}$ cannot be destabilized by any $\mathfrak{S}_k$-invariant subsheaf, which concludes that E_x is $\mu_{\widehat{h}_k}$-stable. □

3.4 A smooth connected component

In this subsection, we will interpret the universal sheaf $\mathcal{E}$ as a family of stable sheaves on $X^{[n]}$ whose base is a smooth connected component of the corresponding moduli space. We have shown above that all the wrong-way fibers E_x of the family $\mathcal{E}$ are μ-stable with respect to $\widehat{h}_k$. We follow the idea in Theorem 2.8 to show their μ-stability with respect to a certain ample class near $\widehat{h}_k$.

Theorem 3.15 *There exists some ample class $H \in \mathrm{NS}(X^{[k]})$ near $\widehat{h}_k$, such that E_x is μ_H-stable for all $x \in X$ simultaneously.*

Proof. The same as in Theorem 2.8, the value of $c = \mu_\beta(E_x)$ is independent of the choice of $x \in X$. We still define

$$S := \{c_1(F) \mid F \subseteq E_x \text{ for some } x \in X \text{ such that } \mu_\beta(F) \geqslant c\}.$$

The proof of the present result is literally the same as the proof of Theorem 2.8, except that the step which shows that S is a finite set has to be modified.

For this purpose we make a few auxiliary definitions. Let $E'_x = \mathcal{G}^\vee \oplus I_{S_x}$ for each $x \in X$. We also define the set

$$S' := \{c_1(F') \mid F' \subseteq E'_x \text{ for some } x \in X \text{ such that } \mu_\beta(F') \geqslant c\}.$$

We claim that $S \subseteq S'$.

Indeed, by (22), every subsheaf $F \subseteq E_x$ is an extension of some subsheaf $F_2 \subseteq I_{S_x}$ by another subsheaf $F_1 \subseteq \mathcal{G}^\vee$. It is then clear that $F' = F_1 \oplus F_2$ is a subsheaf of E'_x, and that $c_1(F) = c_1(F')$. If F destabilizes E_x, then F' also destabilizes E'_x, which means that every element of S is also in S', as desired.

It remains to show that S' is finite. In fact, since $E'_x \subseteq (\mathcal{G}^\vee \oplus \mathcal{O}_{X^{[k]}})$ for all $x \in X$, we obtain that S' is a subset of

$$T' := \{c_1(F') \mid F' \subseteq (\mathcal{G}^\vee \oplus \mathcal{O}_{X^{[k]}}) \text{ such that } \mu_\beta(F') \geqslant c\},$$

which is finite by [7, Theorem 2.29], hence S' is also finite, which further implies the finiteness of S. This concludes the proof. □

Let H be an ample class that satisfies Theorem 3.15, and $\mathcal{M}$ the moduli space of μ_H-stable sheaves on $X^{[k]}$ with the same numerical invariants as E_x. Then the universal family $\mathcal{E}$ defines a classifying morphism

$$f\colon X \longrightarrow \mathcal{M}, \quad x \longmapsto [E_x]. \tag{28}$$

Similar as Theorem 2.10, we obtain

Theorem 3.16 *The classifying morphism* (28) *defined by the family $\mathcal{E}$ identifies X with a smooth connected component of $\mathcal{M}$.*

Proof. For any pair of points $x, y \in X$, since Θ is an equivalence, we have

$$\mathrm{Ext}^*(F_x, F_y) \cong \mathrm{Ext}^*(\mathcal{O}_x, \mathcal{O}_y);$$

moreover by Remark 1.7 we have

$$\mathrm{Ext}^*(E_x, E_y) \cong \mathrm{Ext}^*(F_x, F_y) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{C}).$$

It is clear that

$$\mathrm{Ext}^*_X(\mathcal{O}_x, \mathcal{O}_y) \cong \begin{cases} \Lambda^*(T_{X,x}) & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Combining the above computations we obtain

$$\begin{aligned} \mathrm{Hom}_{X^{[k]}}(E_x, E_y) &= 0 && \text{for any } x, y \in X \text{ with } x \neq y \\ \text{and} \quad \mathrm{Ext}^1_{X^{[k]}}(E_x, E_x) &\cong T_{X,x} && \text{for any } x \in X. \end{aligned}$$

These imply that f is injective on closed points and that $\dim(T_{[E_x]}\mathcal{M}) = 2$ for all $x \in X$. The claim then follows from an argument similar to the proof of Theorem 2.10. □

Remark 3.17 The stable vector bundles constructed in Theorem 2.8 as well as Theorem 3.15 are not tautological bundles as the rank of a tautological bundle is always divisible by k, but in our cases the ranks are $k + 1$ and $2k - 1$.

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References

1. Nicolas Addington. New derived symmetries of some hyperkähler varieties. *Algebr. Geom.*, 3(2):223–260, 2016.
2. Nicolas Addington, Will Donovan, and Ciaran Meachan. Moduli spaces of torsion sheaves on K3 surfaces and derived equivalences. *J. Lond. Math. Soc. (2)*, 93(3):846–865, 2016.
3. Claudio Bartocci, Ugo Bruzzo, and Daniel Hernández Ruipérez. *Fourier-Mukai and Nahm transforms in geometry and mathematical physics*, volume 276 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2009.
4. Mark Andrea A. de Cataldo and Luca Migliorini. The hard Lefschetz theorem and the topology of semismall maps. *Ann. Sci. École Norm. Sup. (4)*, 35(5):759–772, 2002.
5. Olivier Debarre. Hyperkähler manifolds. *arXiv e-prints*, page arXiv:1810.02087, October 2018.
6. Cyril D'Souza. Compactification of generalised Jacobians. *Proc. Indian Acad. Sci. Sect. A Math. Sci.*, 88(5):419–457, 1979.
7. Daniel Greb, Stefan Kebekus, and Thomas Peternell. Movable curves and semistable sheaves. *Int. Math. Res. Not. IMRN*, 2:536–570, 2016.
8. Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.
9. Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.
10. Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.
11. Andreas Krug. Extension groups of tautological sheaves on Hilbert schemes. *J. Algebraic Geom.*, 23(3):571–598, 2014.
12. Andreas Krug. Remarks on the derived McKay correspondence for Hilbert schemes of points and tautological bundles. *Math. Ann.*, 371(1-2):461–486, 2018.
13. Andreas Krug and Pawel Sosna. On the derived category of the Hilbert scheme of points on an Enriques surface. *Selecta Math. (N.S.)*, 21(4):1339–1360, 2015.
14. Andreas Krug and Jørgen Vold Rennemo. Some ways to reconstruct a sheaf from its tautological image on a Hilbert scheme of points. *arXiv e-prints*, 2018. To appear in Math. Nachr.
15. Shigeru Mukai. Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface. *Invent. Math.*, 77(1):101–116, 1984.
16. Shigeru Mukai. Duality of polarized $K3$ surfaces. In *New trends in algebraic geometry (Warwick, 1996)*, volume 264 of *London Math. Soc. Lecture Note Ser.*, pages 311–326. Cambridge Univ. Press, Cambridge, 1999.
17. Fabian Reede and Ziyu Zhang. Examples of smooth components of moduli spaces of stable sheaves. *Manuscripta Math.*, 165(3-4):605–621, 2021.
18. Ulrich Schlickewei. Stability of tautological vector bundles on Hilbert squares of surfaces. *Rend. Semin. Mat. Univ. Padova*, 124:127–138, 2010.
19. Paul Seidel and Richard Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.
20. David Stapleton. Geometry and stability of tautological bundles on Hilbert schemes of points. *Algebra Number Theory*, 10(6):1173–1190, 2016.
21. Malte Wandel. Tautological sheaves: stability, moduli spaces and restrictions to generalised Kummer varieties. *Osaka J. Math.*, 53(4):889–910, 2016.
22. Andrew Wray. *Moduli Spaces of Hermite-Einstein Connections over K3 Surfaces*. PhD thesis, University of Oregon, 2020.
23. Kōta Yoshioka. Some examples of Mukai's reflections on $K3$ surfaces. *J. Reine Angew. Math.*, 515:97–123, 1999.